A method to design tunable quadrature filters in phase shifting interferometry

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Abstract: The main purpose of this paper is to present a method to design tunable quadrature filters in phase shifting interferometry. The algorithm is obtained from a generalized Fourier transform of a symmetrical quadrature filter. This formalism allows us to represent the detuning phase shift error and bias modulation as geometrical conditions. Therefore, the design of the filter becomes a set of solvable linear equations. Hence, to prove our method, several general tunable filters, like three and four frame algorithms, are obtained. Finally, from our results we reproduce particular symmetrical four frame algorithms reported in literature.

OCIS codes: (120.3180) Interferometry; (120.2650) Fringe analysis; (120.5050) Phase Measurement.

References and links

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1. Introduction

Among regular methods to analyze an interferogram, temporal phase shifting (TPS) techniques are considered as one of the most accurate wave-front extraction methods [1–8]. However, the accuracy of these measurements is limited by the presence of several kinds of systematic errors. Two examples are the detuning error due to the miscalibration of the phase shifter and the bias modulation introduced by the use of a source light like a laser diode. In both cases, the errors cannot be eliminated totally, and they may be only minimized. The usual technique to minimize systematic errors is the use of a quadrature filter insensitive to such errors [1–8]. There are several methods to design quadrature filters and a myriad of algorithms exist in the literature [1–8]; however, most of them only give us an algorithm designed for a specific phase step. Hence, in this paper we introduce a simple and practical method to obtain tunable filters. First, the filter is obtained from the general form of the Fourier transform of a quadrature filter. The quadrature conditions of the filter and errors like the detuning error and bias modulation are represented as geometrical conditions to be satisfied by the filter. That is, from the quadrature conditions and the desired errors to

#113584 - $15.00 USD Received 29 Jun 2009; revised 5 Aug 2009; accepted 6 Aug 2009; published 20 Aug 2009

minimize, we obtain a set of solvable linear equations. In other words, from the Fourier transform of the general filter, we apply the geometrical conditions to recover a desired filter. Furthermore, we apply the method to obtain some examples like three and four frame algorithms which are tunable in all cases. Finally, we show that our results for tunable four frame algorithms coincide with particular cases reported in literature, and we also report a table containing the best known ones.

2. Phase-Shifting Interferometry (PSI)

Assuming very low noise, the measured intensity $\mathbf{I}(x,y,t)$ of an interferogram obtained from a CCD detector is expressed as [1–8]

$$\mathbf{I}(x,y,t)=a(x,y)+b(x,y)\cos\left[\varphi\left(x,y\right)+\omega_0 t\right]. \tag{1}$$

Where, $\varphi\left(x,y\right)$ denotes the unknown phase, $a\left(x,y\right)$ is the background illumination, and $b\left(x,y\right)$ is the contrast of interference fringes. The temporal carrier ω_0 is a linear phase shift among the set of interferograms. Meanwhile, *t* corresponds with the temporal sampling, which is taken as a natural number in this paper. To recover the phase in time domain we convolve a set of several temporal phase shifted interferograms with a quadrature filter that has a temporal impulse response $h(t)$ [7–9], and it yields an output signal $g(t)$ given by $g(t)=h(t)*\mathbf{I}(x,y,t)$ where (*) denotes one dimensional temporal convolution. Therefore, in the Fourier domain we have that this temporal convolution becomes $G(\omega)=H(\omega)I(\omega)$, where $I(\omega)$ and $G(\omega)$ are given by

$$I(\omega)=2a\pi\delta\left(\omega\right)+b\pi\exp\left(-i\varphi\right)\delta\left(\omega+\omega_0\right)+b\pi\exp\left(i\varphi\right)\delta\left(\omega-\omega_0\right). \tag{2}$$

$$G(\omega)=2a\pi\, H(\omega)\delta(\omega)+b\pi\, H(\omega)\exp\left(-i\varphi\right)\delta(\omega+\omega_0)+b\pi\, H(\omega)\exp\left(i\varphi\right)\delta(\omega-\omega_0). \tag{3}$$

Notice that the spatial dependence (*x,y*) of the functions *I*, *a*, *b* and φ has been dropped.

Now, considering a symmetric quadrature filter, the Fourier transform $H(\omega)$ becomes a real function [9]. Then, assuming that the quadrature filter is tuned onto the right side of the frequency axis, as depicted in Fig. (1) by the solid curve, at exactly the carrier frequency $\omega=-\omega_0$, we have the quadrature conditions as $H(\omega_0)=0$ and $H(0)=0$, while $H(-\omega_0)\neq 0$. Therefore, from Eq. (3) output signal $G(\omega)$ becomes

$$G(-\omega_0)=b\pi\, H(-\omega_0)\exp(-i\varphi). \tag{4}$$

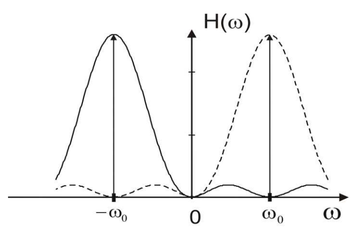

Fig. 1. Graphical representation of a quadrature filter

That is, the estimated phase for a filter tuned onto the right side is given by $-\varphi$ or $\pi-\varphi$.

Another possibility to recover the desired phase is to find a quadrature filter tuned onto the left side of the frequency axis. Then; as shown in Fig. (1) by the dashed curve, at exactly the carrier frequency $\omega=\omega_0$ with $H(\omega_0)\neq 0$, the quadrature conditions are given by $H(-\omega_0)=0$ and $H(0)=0$. Therefore, output signal $G(\omega)$ becomes,

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$$G(\omega_0) = b\pi\, H(\omega_0)\exp(i\varphi). \quad (5)$$

Then, the estimated phase is ϕ. Thus, we can conclude that the design of a quadrature filter can be viewed as a geometrical design problem, where the interception with the frequency axis gives us the quadrature conditions or the cancellation of a specific harmonic frequency. In the same way, the derivate that has a value of zero at a specific frequency gives us the condition for a quadrature filter insensitive to linear phase shift error at that frequency. Finally, when the derivate equals zero at the origin, we have the conditions for a filter which is insensitive to bias variation error [2–5]. Therefore, assuming a symmetrical algorithm with order N and tuned onto the left side of the frequency axis, the estimated phase is

$$\tan(\varphi) = \frac{\sum_{k=1}^{N} b_k I_k}{\sum_{k=1}^{N} a_k I_k}. \quad (6)$$

where the coefficients a_k and b_k for symmetrical algorithms satisfy the relationships $b_k = -b_{N-k+1}$ and $a_k = a_{N-k+1}$. Notice that for odd N, the middle term is $b_{(N+1)/2} = 0$ [7,8]. Then, the temporal impulse response [2–9] of this filter $h(t)$ is given by

$$h(t) = \sum_{k=1}^{N} a_k \left[\delta(t-p)\right] + i\sum_{k=1}^{N} b_k \left[\delta(t-p)\right]. \quad (7)$$

where $p = (2k - N - 1)/2$ is chosen to obtain a symmetrical filter. Since Eq. (7) corresponds with a quadrature filter, the real part must be an even function, whereas the imaginary part is an odd function. Then, the Fourier transform of $h(t)$ generates the real function $H(\omega)$, given by

$$H(\omega) = \sum_{k=1}^{N/2} a_k \cos(p\omega) - \sum_{k=1}^{N/2} b_k \sin(p\omega); \text{ For even } N. \quad (8)$$

$$H(\omega) = \frac{a_{(N+1)/2}}{2} + \sum_{k=1}^{(N-1)/2} a_k \cos(p\omega) - \sum_{k=1}^{(N-1)/2} b_k \sin(p\omega); \text{ For odd } N \quad (9)$$

From Eq. (6) we can observe that to design a desired quadrature filter with N frames as support, only N-1 parameters are independent. Then, taking a step α, the geometrical conditions to satisfy a quadrature filter tuned onto the left side are given by

$$H(0) = 0;\ H(-\alpha) = 0. \quad (10)$$

That is, the filter cuts-off both $\omega = 0$ and $\omega = -\alpha$ frequencies. Then, the number of independent parameters to design the filter is reduced to three, like in the three frame algorithm case. However, if filter order N becomes greater than three, we can use these other conditions to eliminate or compensate some kinds of errors, like detuning phase shift or bias variation errors. That is, to design a filter tuned onto the left side and insensitive to the n^{th} order detuning phase shift error, we just apply the following geometrical conditions,

$$H'(-\alpha) = 0;\ H''(-\alpha) = 0;\ H'''(-\alpha) = 0\ \ldots\ H^{n}(-\alpha) = 0. \quad (11)$$

Where, n gives us the order of insensitivity to this phase shift error, and $H^{n}(\omega)$ is the n^{th} derivate of $H(\omega)$ with regard to ω.

On the other hand, to design a filter that is also insensitive to the m^{th} order bias variation error, we can add the following conditions

$$H'(0) = 0;\ H''(0) = 0;\ H'''(0) = 0\ \ldots\ H^{m}(0) = 0. \quad (12)$$

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Hence, to obtain a high order of insensitivity to one error, like detuning or bias variation, we simply increase the order of derivate conditions. That is, the maximum number of independent conditions to design a quadrature filter becomes N-3. Then, we can decide the conditions to apply and compensate for some kind of errors like the detuning phase shift error or bias modulation, or simply cancel a specific frequency. These conditions always give us a set of solvable linear equations whose solution produces the coefficients needed for the filter.

In conclusion, the design of a tunable quadrature filter is a geometrical problem where the design conditions are interpreted as the interception over frequencies $\omega = 0$ and $\omega = -\alpha$ of $H(\omega)$, while the insensitivity to some errors, like phase shift and bias modulation, is interpreted as the n derivates equal to zero. Finally, the problem is reduced to solving a set of linear equations and recovering from Eq. (6) the desired TPS algorithm.

3. Designing some examples of a tunable quadrature filter

The above method was applied to two known algorithms, one odd and another even.

3.1 The tunable three frame algorithm

From Eq. (6), the general symmetrical form of a three frame algorithm is,

$$\tan(\varphi) = \frac{b_1 I_1 - b_1 I_3}{a_1 I_1 + a_2 I_2 + a_1 I_3}. \tag{13}$$

As mentioned above, in Eq. (13) only two parameters are independent, Therefore, by simplicity for N odd we choose $a_2 = 2$. Then, from Eq. (9) Fourier transform of Eq. (13) for a step α given by Eq. (9) is,

$$H(\omega) = 1 + a_1 \cos(\omega) - b_1 \sin(\omega). \tag{14}$$

Tuning this filter onto the left side, it must satisfy the quadrature conditions given by Eq. (10). Then, $H(\omega)$ satisfies $H(0) = H(-\alpha) = 0$, and from Eq. (14) we obtain the linear system

$$H(0) = 1 + a_1 = 0. \tag{15}$$

$$H(-\alpha) = 1 + a_1 \cos(\alpha) + b_1 \sin(\alpha) = 0. \tag{16}$$

Therefore, the from Eq. (15) $a_1 = -1$ and from Eq. (16)

$$b_1 = -\left[1 - \cos(\alpha)\right] / \sin(\alpha). \tag{17}$$

Then, from Eq. (13) we obtain the tunable three frame algorithm as,

$$\tan(\varphi) = \left[\frac{1-\cos(\alpha)}{\sin(\alpha)}\right] \frac{I_1 - I_3}{I_1 - 2I_2 + I_3} = \tan(\alpha/2)\left(\frac{I_1 - I_3}{I_1 - 2I_2 + I_3}\right). \tag{18}$$

It should be noticed that choosing a quadrature filter tuned onto the right side of the frequency axis we obtain the same result. Notice that using another value non zero for a_2 the same result Eq. (18) is obtained.

#113584 - $15.00 USD Received 29 Jun 2009; revised 5 Aug 2009; accepted 6 Aug 2009; published 20 Aug 2009

3.2 The tunable four frame algorithms

To design a four fame (TPS) algorithm we have only one independent parameter used to minimize one kind of error. Therefore, we have only three options; the first one to obtain a filter equal to the least squares method, or class A filter. The second one corresponds to a filter that is insensitive to the linear detuning phase shift error. Finally, the last option is for a filter that is insensitive to the linear bias variation error.

3.2.3 Tunable four frame filter class A

From Eq. (6) we have that the general form for a four frame algorithm tuned onto right is

$$\tan(\varphi) = \frac{b_1 I_1 + b_2 I_2 - b_2 I_3 - b_1 I_4}{a_1 I_1 + a_2 I_2 + a_2 I_3 + a_1 I_4}. \tag{19}$$

Then, from Eq. (8) and taking $a_2 = 1$, the Fourier transform of the filter becomes

$$H(\omega) = a_1 \cos(3\omega/2) + \cos(\omega/2) - b_1 \sin(3\omega/2) - b_2 \sin(\omega/2). \tag{20}$$

Then, to obtain an algorithm through the least squares method, geometrical conditions are

$$H(0) = H(-\alpha) = H'(\alpha) = 0. \tag{21}$$

That is, the two conditions given by Eq. (10) are the quadrature conditions that any quadrature filter must satisfy, and $H'(\alpha) = 0$ is the particular condition of this filter to be insensitive to linear variations of the value $H(\alpha)$. Then, applying Eq. (20) into Eq. (21) we have

$$H(0) = 1 + a_1 = 0. \tag{22}$$

$$H(-\alpha) = \cos(\alpha/2) + a_1 \cos(3\alpha/2) + b_1 \sin(3\alpha/2) + b_2 \sin(\alpha/2) = 0. \tag{23}$$

$$2H'(\alpha) = -3\sin(\alpha/2) - 3a_1 \sin(3\alpha/2) - 3b_1 \cos(3\alpha/2) - b_2 \cos(\alpha/2) = 0. \tag{24}$$

And the solution for this filter tuned onto the right of the frequency axis is given by,

$$a_1 = -1; \quad b_1 = \left[1 + 2\cos(\alpha)\right] / \sin(\alpha); \quad b_2 = -\left[2 + 4\cos(\alpha) + 3\cos(2\alpha)\right] / \sin(\alpha). \tag{25}$$

Then, by substituting into Eq. (19), we obtain our tunable four frame algorithm as

$$\tan(\varphi) = \frac{-\left[1 + 2\cos(\alpha)\right](I_1 - I_4) + \left[2 + 4\cos(\alpha) + 3\cos(2\alpha)\right](I_2 - I_3)}{\sin(\alpha)(I_1 - I_2 - I_3 + I_4)}. \tag{26}$$

According to the authors in [5], this filter can be called a tunable four frame filter class A. In a previous work [9], we present a formalism to obtain the detuning error of a symmetrical filter. Then, for a small detuning phase Δ, the detuning error $\Delta\phi$ of this quadrature filter is

$$\Delta\varphi = -\tan(\Delta/2) \approx -\Delta/2. \tag{27}$$

As we expect, this algorithm is very sensitive to errors like detuning or bias modulation.

3.2.2 Tunable four frame algorithm insensitive to linear detuning phase shift error

From Eq. (11), a filter that is insensitive to the linear detuning error satisfies the condition

$$2H'(\alpha) = -3\sin(\alpha/2) - 3a_1 \sin(3\alpha/2) + 3b_1 \cos(3\alpha/2) + b_2 \cos(\alpha/2) = 0. \tag{28}$$

Then, by using the quadrature conditions Eq. (22) and Eq. (23) with Eq. (28), our solution is

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$$a_1 = -1; \quad b_1 = -\cos(\alpha)/\sin(\alpha); \quad b_2 = \left[2+\cos(\alpha)\right]/\sin(\alpha). \tag{29}$$

Therefore, the tunable algorithm that is insensitive to linear detuning phase shift error is

$$\tan(\varphi) = \frac{\cos(\alpha)(I_1 - I_4) - \left[2+\cos(\alpha)\right](I_2 - I_3)}{\sin(\alpha)(I_1 - I_2 - I_3 + I_4)}. \tag{30}$$

This filter, according to [5], can be called a tunable filter class B. Then from [9], for small phase detuning, the detuning error calculated is $\Delta\varphi = \tan^2(\Delta/2) \approx \Delta^2/4$. That is, as we expect, the algorithm is insensitive to the linear detuning phase shift error.

3.2.3 Tunable four frame algorithm insensitive to linear bias variation error

From Eq. (12) to obtain a filter that is insensitive to the linear bias variation error, $H(\omega)$ must satisfy the following geometrical condition

$$2H'(0) = -3b_1 - b_2 = 0. \tag{31}$$

Then, combining the quadrature conditions Eq. (22) and Eq. (23) with Eq. (31), we obtain

$$a_1 = -1; \quad b_1 = -1/\tan(\alpha/2); \quad b_2 = 3/\tan(\alpha/2). \tag{32}$$

And from Eq. (20), the tunable four frame filter insensitive to linear bias variation error is

$$\tan(\varphi) = \frac{1}{\tan(\alpha/2)}\left(\frac{I_1 - 3I_2 + 3I_3 - I_4}{I_1 - I_2 - I_3 + I_4}\right). \tag{33}$$

Therefore, this algorithm can be called a tunable filter class C. Then, from [9], the detuning error is $\Delta\phi = \tan(\Delta/2)$. Finally, in Table 1, from our solutions showed above, we reproduce particular cases reported in literature [5–8].

Table 1. Several particular four frame TPS algorithms class A, B and C

#	step	tan(ϕ) Class A	tan(ϕ) Class B	Tan(ϕ) Class C
1	π/4	$-(2+\sqrt{2})\frac{I_1-2I_2+2I_3-I_4}{I_1-I_2-I_3+I_4}$	$\frac{I_1-(2\sqrt{2}+1)(I_2-I_3)-I_4}{I_1-I_2-I_3+I_4}$	$(\sqrt{2}+1)\left(\frac{I_1-3I_2+3I_3-I_4}{I_1-I_2-I_3+I_4}\right)$
2	π/3	$-\frac{4I_1-5I_2+5I_3-4I_4}{\sqrt{3}(I_1-I_2-I_3+I_4)}$	$\frac{I_1-5I_2+5I_3-I_4}{\sqrt{3}(I_1-I_2-I_3+I_4)}$	$\frac{\sqrt{3}(I_1-3I_2+3I_3-I_4)}{I_1-I_2-I_3+I_4}$
3	π/2	$-\frac{I_1+I_2-I_3-I_4}{I_1-I_2-I_3+I_4}$	$\frac{-2(I_2-I_3)}{I_1-I_2-I_3+I_4}$	$\frac{I_1-3I_2+3I_3-I_4}{I_1-I_2-I_3+I_4}$
4	2π/3	$\frac{-\sqrt{3}(I_2-I_3)}{I_1-I_2-I_3+I_4}$	$\frac{-I_1-3I_2+3I_3+I_4}{\sqrt{3}(I_1-I_2-I_3+I_4)}$	$\frac{I_1-3I_2+3I_3-I_4}{\sqrt{3}(I_1-I_2-I_3+I_4)}$

4. Conclusions

An exact, analytical algorithm to design quadrature filters as a geometrical problem based on the Fourier description of the quadrature filter is presented. The quadrature conditions and the error are interpreted geometrically, and the problem is reduced to solving a set of linear equations. Then, from such solution we can obtain tunable filters with any combination of error, instead of using other known procedures [1–8]; foremost, we apply the method to obtain some tunable filters for three and four frame cases. From our solutions, we generalize all four frame symmetrical algorithm cases reported in literature, and a table with some particular cases is reported.

#113584 - $15.00 USD Received 29 Jun 2009; revised 5 Aug 2009; accepted 6 Aug 2009; published 20 Aug 2009