# Calculus of exact detuning phase shift error in temporal phase shifting algorithms 

J. F. Mosiño, ${ }^{1, *}$ D. Malacara Doblado, ${ }^{1}$ and D. Malacara Hernández ${ }^{\mathbf{1}}$<br>${ }^{1}$ Centro de Investigaciones en Óptica, Loma del Bosque 115, A. P. 1-948, León, Gto. 20036, México *jfmosino@cio.mx


#### Abstract

The detuning phase shift error is a common systematic error observed in temporal phase shifting (TPS) algorithms. This error, generally due to miscalibration of the phase shifter, is solved by using a quadrature filter insensitive to this detuning error. To compare algorithms, this error is frequently analyzed numerically. However, in this work we present an exact and analytical expression to calculate such error which is applicable to any kind of filters with real or complex frequency response. Finally, a table with the detuning error for several algorithms is reported.


©2009 Optical Society of America
OCIS codes: (120.3180) Interferometry; (120.2650) Fringe analysis; (120.5050) Phase Measurement.

## References and links

1. J. Schwider, "Advanced evaluation techniques in interferometry," in Progress in Optics, E. Wolf ed., (North Holland, Amsterdam, Oxford, New York, Tokyo, 1990).
2. J. H. Bruning, D. R. Herriott, J. E. Gallagher, D. P. Rosenfeld, A. D. White, and D. J. Brangaccio, "Digital wavefront measuring interferometry for testing optical surfaces and lenses," Appl. Opt. 13(11), 2693-2703 (1974).
3. H. Schreiber, J. H. Brunning, and J. E. Greivenkamp, "Phase shifting interferometry," in Optical Shop Testing, D. Malacara ed., (John Wiley \& Sons, Inc., Hoboken, New Jersey 2007).
4. M. Servin, and M. Kujawinska, "Modern fringe pattern analysis in Interferometry," in Handbook of Optical Engineering, D. Malacara and B. J. Thompson eds., (Marcel Dekker, 2001).
5. D. Malacara, M. Servin, and Z. Malacara, "Phase Detection Algorithms," in Interferogram Analysis for Optical Testing, D. Malacara ed., (Taylor \& Francis Group, 2005).
6. J. Schwider, R. Burow, K. E. Elssner, J. Grzanna, R. Spolaczyk, and K. Merkel, "Digital wave-front measuring interferometry: some systematic error sources," Appl. Opt. 22(21), 3421-3432 (1983).
7. K. Freischlad, and C. L. Koliopoulos, "Fourier description of digital phase measuring interferometry," J. Opt. Soc. Am. A 7(4), 542-551 (1990).
8. J. E. Hernández and D. Malacara, "Exact linear detuning error in phase shifting algorithms," Opt. comm. 180, 914 (2000).
9. J. F. Mosiño, M. Servin, J. C. Estrada, and J. A. Quiroga, "Phasorial analysis of detuning error in temporal phase shifting algorithms," Opt. Express 17(7), 5618-5623 (2009).

## 1. Introduction

Temporal Phase Shifting (TPS) techniques are used widely for wave front extraction [1-8]. However, the accuracy of these measurements is limited by the presence of several systematic errors. One important error to estimate is the detuning phase shift error, which is due to a miscalibration of the phase shifter and it is introduced by the data gathering process. To minimize the error due to detuning, the temporal signal's carrier $\omega_{0}$ must have exactly the carrier frequency assumed in the TPS algorithm; otherwise, an erroneous phase is estimated. In a previous work [9], we obtained an exact expression by using a phasorial method to analytically calculate the detuning phase shift error of the TPS algorithm. However, this demonstration is only valid for symmetrical TPS algorithms having a real frequency response. Although, in ref [5] the author show how a non symmetrical filter can be transformed into a symmetric filter, the main purpose of this paper is to find a general useful expression to calculate the detuning phase shift error in terms of the frequency response of any TPS algorithm, symmetrical or not without the use of any transformation for the filter. This paper is organized as follows: in Section 2 we discuss the fact that a detuning phase shift error is a
very common systematic error, and we obtain a generalized expression to calculate this error. In Section 3, some particular cases of non symmetrical TPS algorithms are considered, as three and four frames cases, and a table of analytical expressions of detuning errors for several known algorithms is reported. Finally in Section 4, the conclusions are discussed.

## 2. Error Detuning in Phase-Shifting Interferometry (PSI)

The measured intensity of an interferogram on a CCD detector can be expressed by [3,4]:

$$
\begin{equation*}
\mathbf{I}\left(x, y, t, \omega_{0}\right)=a(x, y)+b(x, y) \cos \left[\phi(x, y)+\omega_{0} t\right] . \tag{1}
\end{equation*}
$$

Where, $\phi(x, y)$ denotes the unknown phase, $a(x, y)$ is the background illumination, and $b(x, y)$ is the contrast of interference fringes; these two signals are low frequency. The temporal carrier $\omega_{0}$ is a linear phase shift among the set of interferograms that is introduced in the data gathering process. Meanwhile, $t$ corresponds with the temporal sampling, which is taken as a natural number in this paper. Taking the Fourier transform of $\mathbf{I}(x, y, t)$, we have

$$
\begin{equation*}
I(\omega)=a \delta(\omega)+\pi b \exp (i \phi) \delta\left(\omega-\omega_{0}\right)+\pi b \exp (-i \phi) \delta\left(\omega+\omega_{0}\right) \tag{2}
\end{equation*}
$$

where, for the sake of simplicity, the spatial dependence $(x, y)$ of the functions $I, a, b$ and $\phi$ has been dropped. Then, the desired phase $\phi$ is obtained convolving a discrete temporal quadrature filter $h(t)$ with several temporal phase shifted interferograms to obtain an output function $g(t)$, described as

$$
\begin{equation*}
g(t)=h(t) * I(x, y, t) \tag{3}
\end{equation*}
$$

where $\left({ }^{*}\right)$ denotes the one dimensional temporal convolution. Then, the phase is recovered from this output signal. Taking the Fourier transform of Eq. (3), we have $G(\omega)=I(\omega) H(\omega)$, and it can be expressed as

$$
\begin{equation*}
G(\omega)=a H(\omega) \delta(\omega)+\pi b H(\omega) \exp (i \phi) \delta\left(\omega-\omega_{0}\right)+\pi b H(\omega) \exp (-i \phi) \delta\left(\omega+\omega_{0}\right) \tag{4}
\end{equation*}
$$

By using the phasor form $H(\omega)=|H(\omega)| \exp [i \theta(\omega)]$, where $|H(\omega)|$ satisfies the quadrature conditions, $\theta(\omega)$ must be an even or odd function, then output $G(\omega)$ is

$$
\begin{equation*}
G(\omega)=a H(\omega) \delta(\omega)+\pi b|H(\omega)| \exp (i \phi+i \theta) \delta\left(\omega-\omega_{0}\right)+\pi b|H(\omega)| \exp (-i \phi+i \theta) \delta\left(\omega+\omega_{0}\right) . \tag{5}
\end{equation*}
$$

Now, to recover the phase from Eq. (5), we have three components, and to obtain a quadrature filter we only have two possible options, as long as $\omega_{0}>0$. The first one is the case for a filter tuned onto the right side of the frequency axis, which for frequencies $\omega=0$ and $\omega=\omega_{0}$, meets the condition $|H(0)|=\left|H\left(\omega_{0}\right)\right|=0$, while $\left|H\left(-\omega_{0}\right)\right| \neq 0$. The second option is for a filter tuned onto the left side of frequency axis, and the quadrature conditions for the frequencies $\omega=0$ and $\omega=-\omega_{0}$ are $|H(0)|=\left|H\left(-\omega_{0}\right)\right|=0$ with $\left|H\left(\omega_{0}\right)\right| \neq 0$. That is, the quadrature conditions are given only by the magnitude of the filter. Then, there are two possible solutions to recover the desired phase. Therefore, for a quadrature filter tuned onto the right side case, the output of the TPS algorithm, $G\left(\omega_{0}\right)$ becomes

$$
\begin{equation*}
G\left(\omega_{0}\right)=\pi b\left|H\left(-\omega_{0}\right)\right| \exp \left[-i \phi+\theta\left(-\omega_{0}\right)\right] \tag{6}
\end{equation*}
$$

then, the recovered phase is $-\phi \pm \theta\left(\omega_{0}\right)$. On the other hand, for the left tuned case,

$$
\begin{equation*}
G\left(-\omega_{0}\right)=\pi b\left|H\left(\omega_{0}\right)\right| \exp \left[i \phi+\theta\left(\omega_{0}\right)\right] \tag{7}
\end{equation*}
$$

And the recovered phase is $\phi+\theta\left(\omega_{0}\right)$. That is, to recover the desired phase with a sign matching that of the real phase, we must choose a quadrature filter tuned onto the left side of frequency axis; otherwise, we get the same phase, but with opposite sign. However, for a frequency $\omega=-\omega_{0}-\Delta$ we have a detuned output $G\left(-\omega_{0}-\Delta\right)$ as

$$
\begin{equation*}
G\left(-\omega_{0}-\Delta\right)=\exp \left[i \theta\left(\omega_{0}+\Delta\right)\right]\{c \exp (i \phi) \pm \varepsilon \exp (-i \phi)\} \tag{8}
\end{equation*}
$$

where constants $c$ and $\varepsilon$ can be observed in Fig. 1 and are given by,

$$
\begin{equation*}
c=\pi b\left|H\left(\omega_{0}+\Delta\right)\right| ; \varepsilon=\pi b\left|H\left(-\omega_{0}-\Delta\right)\right| . \tag{9}
\end{equation*}
$$



Fig. 1. Detuned components $c$ and $\varepsilon$.
Then, taking just the sign plus in Eq. (8) and rearranging $G\left(-\omega_{0}-\Delta\right)$ we have

$$
\begin{equation*}
G\left(-\omega_{0}-\Delta\right)=\exp \left[i \theta\left(\omega_{0}+\Delta\right)\right]\{\cos (\phi)(c+\varepsilon)+i \sin (\phi)(c-\varepsilon)\} \tag{10}
\end{equation*}
$$

Then, the erroneous phase $\phi^{\prime}$ from Eq. (10) is

$$
\begin{equation*}
\tan \left(\phi^{\prime}\right)=\frac{c-\varepsilon}{c+\varepsilon} \tan (\phi)=\sigma \tan (\phi) \tag{11}
\end{equation*}
$$

Notice that the right side of Eq. (11) has been reported previously in [7,8], where, $\sigma$ is the correlation factor. In the same way, $r$ is the ratio of detuning, and the relationship between both values is given by

$$
\begin{equation*}
r=\frac{\varepsilon}{c}=\frac{1-\sigma}{1+\sigma}=\frac{\left|H\left(-\omega_{0}-\Delta\right)\right|}{\left|H\left(+\omega_{0}+\Delta\right)\right|} \tag{12}
\end{equation*}
$$

Then, the detuning error $\Delta \phi$ may be defined as the difference between the desired phase $\phi$ and the undesired phase $\phi^{\prime}$ as $\Delta \phi=\phi^{\prime}-\phi$. This expression is widely used to numerically evaluate the detuning error in TPS algorithms [1-8]. Substituting Eq. (11) into the definition of detuning error, we find that this error is expressed as,

$$
\begin{equation*}
\Delta \phi=\tan ^{-1}[\sigma \tan (\phi)]-\phi \tag{13}
\end{equation*}
$$

By using a different method, in $\operatorname{Ref}[7,8]$. the authors find the same equation, but they are only able to calculate an approximation for it. In a previous work [9], by using a phasorial method,
we find an expression for the detuning error applicable to symmetrical filters as a function of ratio $r$. Here, we expand our previous work to both symmetrical and non-symmetrical quadrature filters, and additionally, we establish the relationship between both formalisms.

To solve Eq. (13), we take the tangent for both terms, and we have,

$$
\begin{equation*}
\tan (\Delta \phi)=-\frac{\tan (\phi)-\sigma \tan (\phi)}{1+\sigma \tan ^{2}(\phi)} \tag{14}
\end{equation*}
$$

This expression can be rearranged to obtain,

$$
\begin{equation*}
\tan (\Delta \phi)=-\frac{\left(\frac{1-\sigma}{1+\sigma}\right) \frac{2 \tan (\phi)}{1+\tan ^{2}(\phi)}}{1+\left(\frac{1-\sigma}{1+\sigma}\right)\left(\frac{1-\tan ^{2}(\phi)}{1+\tan ^{2}(\phi)}\right)} \tag{15}
\end{equation*}
$$

Then, by using Eq. (11) and after some trigonometric substitutions, Eq. (14) becomes,

$$
\begin{equation*}
\tan (\Delta \phi)=-\frac{r \sin (2 \phi)}{1+r \cos (2 \phi)} \tag{16}
\end{equation*}
$$

Although this analytical expression is similar to the equation previously reported [8], the ratio reported in this work is more general than the previously reported ratio, in spite of both being referred to as $r$. That is, from Eq. (15) and the ratio $r$ we can obtain the exact detuning error for any TPS algorithm. Then, from Eq. (15) we can observe that if $|\varepsilon| \rightarrow 0$, then $r \rightarrow 0$ and $\sigma \rightarrow 1$, no detuning error is present, and the erroneous phase $\phi^{\prime}$ becomes the desired phase $\phi^{\prime} \rightarrow \phi$. On the other hand, assuming a small detuning error, we have that $\tan (\Delta \phi) \approx \Delta \phi$ and $r \cos (2 \phi) \ll 1$. Hence, Eq. (16) is reduced to,

$$
\begin{equation*}
\Delta \phi=-r \sin (2 \phi)=-\frac{\left|H\left(-\omega_{0}-\Delta\right)\right|}{\left|H\left(\omega_{0}+\Delta\right)\right|} \sin (2 \phi)=-\left(\frac{\sigma-1}{\sigma+1}\right) \sin (2 \phi) \tag{17}
\end{equation*}
$$

This expression may be further simplified by using $\sigma \approx 1.0$. Doing this, we recover the expression $\Delta \phi \approx 0.5(\sigma-1) \sin (2 \phi)$, which was reported in literature [7,8]. Notice that this result is almost the same result presented here; however, we consider the expression here reported to be more practical. To compare against the results reported in literature, we maximize our exact result in Eq. (17) with respect to $\phi$, obtaining the following expression,

$$
\begin{equation*}
\left|\Delta \phi_{\max }\right|=\sin ^{-1}\left|H\left(-\omega_{0}-\Delta\right) / H\left(\omega_{0}+\Delta\right)\right| \tag{18}
\end{equation*}
$$

We emphasize that this Eq. (18) for the maximum detuning error is exact when tuned onto the left side; in this fashion, it coincides exactly with the detuning error that was evaluated numerically [1-7]. We can repeat all the steps described above for a quadrature filter tuned onto the right side of the frequency axis, or for sign minus in Eq. (8) and we obtain the following result, which is equivalent to having changed the sign of $\omega_{0}+\Delta$, then, we have

$$
\begin{equation*}
\left|\Delta \phi_{\max }\right|=\sin ^{-1}\left|H\left(\omega_{0}+\Delta\right) / H\left(-\omega_{0}-\Delta\right)\right| . \tag{19}
\end{equation*}
$$

In consequence, it can be said that the user must take whether the quadrature filter is tuned onto the left or onto the right before applying the formula. Finally, we must notice that for a symmetrical filter, the frequency response becomes a real function and the result coincides with what has been previously reported [9]. This expression is a very versatile way to evaluate
the detuning phase shift error analytically or numerically, instead of the approximation reported in literature $[7,8]$.

## 3. Some Examples of Error Detuning in Phase-Shifting Interferometry

In this Section we analyze some popular TPS algorithms, such as the three and four frame cases.
3.1 Three frame algorithm case

One three frame non symmetric TPS algorithm is given by

$$
\begin{equation*}
\tan [\phi(x, y, \alpha=\pi / 2)]=\frac{I(-\alpha)-I(0)}{I(0)-I(\alpha)} \tag{20}
\end{equation*}
$$

The time response of this quadrature filter with $\alpha=\pi / 2$ is,

$$
\begin{equation*}
h(t, \alpha=\pi / 2)=[\delta(t)-\delta(t-\alpha)]+i[\delta(t+\alpha)-\delta(t)] \tag{21}
\end{equation*}
$$

The frequency response becomes non-real; then, $H(\omega, \alpha)$ is

$$
\begin{equation*}
H(\omega, \alpha)=i \exp [\omega \alpha i]-\exp [-\omega \alpha i]+1-i ; \quad \alpha=\pi / 2 \tag{22}
\end{equation*}
$$

This filter is tuned at frequency $\omega=1$ with $\alpha=\pi / 2$. That is, the filter satisfies $H(\omega=1, \alpha=\pi / 2)=H(\omega=0, \alpha=\pi / 2)=0$, meaning that it is tuned onto the right side.
Now, from Eq. (18), the exact detuning error for $\alpha=\pi / 2+\Delta$ is

$$
\begin{equation*}
\Delta \phi_{\max }=\sin ^{-1}\left|\frac{H(\omega=1, \alpha=\pi / 2+\Delta)}{H(\omega=1, \alpha=-\pi / 2-\Delta)}\right| . \tag{23}
\end{equation*}
$$

Then, calculating this ratio we have

$$
\begin{equation*}
\Delta \phi_{\max }=\sin ^{-1}\left|\frac{-2 i(i+1) \sin (\Delta / 2)[\sin (\Delta / 2)+\cos (\Delta / 2)]}{-2 i(i-1) \cos (\Delta / 2)[\sin (\Delta / 2)+\cos (\Delta / 2)]}\right|=\sin ^{-1}\left|\tan \left(\frac{\Delta}{2}\right)\right| \approx|\Delta| / 2 . \tag{24}
\end{equation*}
$$

### 3.2 Four frame algorithm

The four non symmetric frame TPS in cross algorithm is given by

$$
\begin{equation*}
\phi(x, y, \alpha=\pi / 2)=\tan ^{-1}\left(\frac{I(-\alpha)-I(\alpha)}{I(0)-I(2 \alpha)}\right) \tag{25}
\end{equation*}
$$

Then, the frequency response of this TPS algorithm is a complex function given by,

$$
\begin{equation*}
H(\omega, \alpha)=1-2 \sin (\omega \alpha)-\exp (-2 \omega \alpha i) \tag{26}
\end{equation*}
$$

Notice that this filter is also tuned onto the right side at $\omega=1$ with a phase step $\alpha=\pi / 2$, and we have that $H(\omega=1, \alpha=\pi / 2)=0$ and $H(\omega=0, \alpha=\pi / 2)=0$. Now, from Eq. (19) we obtain,

$$
\begin{equation*}
\Delta \phi_{\max }=\sin ^{-1}|i \exp (-\Delta i) \tan (\Delta / 2)|=\sin ^{-1}|\tan (\Delta / 2)| \approx|\Delta| / 2 \tag{27}
\end{equation*}
$$

### 3.2 Other algorithms

In Table 1, the value $r$, for some detuning phase shift errors for several TPS algorithms are presented. We notice that many of them have the form $\tan ^{n}(\Delta / 2)$ for $n$ integer.

## 4. Conclusions

An exact and analytical algorithm to evaluate the detuning error in phase shifting algorithms was obtained from algebraic methods. The expression is applicable to any kind of (PSI) algorithms, symmetrical or not. The derived expression was compared with other well known approximations. Finally, this expression was successfully applied to evaluate and obtain the detuning error for some well known quadrature filters.

Table 1. Detuning Phase Shift error for several TPS algorithms

| \# | N | Step | $\tan (\phi)=$ Num/Den | $r$ | Ref |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 3 | $\pi / 2$ | [1,-1,0]/ [0,1,-1] | $-\tan (\Delta / 2)$ | 5 |
| 2 | 3 | $\pi / 2$ | [1,-2,1]/ [1,0,-1] | $-\tan (\Delta / 2)$ | 3 |
| 3 | 4 | $\pi / 2$ | [1,1,-1,-1]/[-1,1,1,-1] | $-\tan (\Delta / 2)$ | 3 |
| 4 | 4 | $\pi / 2$ | [0,-1,0,1]/[1,0,-1,0] | $-\tan (\Delta / 2)$ | 3 |
| 5 | 4 | $\pi / 2$ | [0,-2,2,0]/ [1,-1,-1,1] | $\tan ^{2}(\Delta / 2)$ | 5 |
| 6 | 4 | $\pi / 2$ | [1,-3, 1,1]/ [1,1,-3,1] | $\tan ^{2}(\Delta / 2)$ | 5 |
| 7 | 5 | $\pi / 2$ | [0,-2,0,2,0]/[1,0,-2,0,1] | $\tan ^{2}(\Delta / 2)$ | 7 |
| 8 | 5 | $\pi / 2$ | [0,-3,3,1,1]/ [1,-1,-3,3,0] | $-\tan ^{3}(\Delta / 2)$ | 5 |
| 9 | 5 | $\pi / 2$ | [-1,2,0,-2,1]/[0,-2,4,2,0] | $-\tan ^{2}(\Delta / 2)$ | 5 |
| 10 | 6 | $\pi / 2$ | [1,-5,-2,10,-3,-1]/ [1,3,-10,2,5,-1] | $\tan ^{4}(\Delta / 2)$ | 5 |
| 11 | 6 | $\pi / 2$ | [0,-3,0,4,0,-1]/[1,0,-4,0,3,0] | $-\tan ^{3}(\Delta / 2)$ | 5 |
| 12 | 6 | $\pi / 2$ | [0,-4,4,4,-4,0]/ [1,-1,-6,6,1,-1] | $\tan ^{4}(\Delta / 2)$ | 5 |
| 13 | 6 | $\pi / 2$ | [0,-2,-2,2,2,0]/ [1,1,-2,-2,1,1] | $\tan ^{2}(\Delta / 2)$ | 5 |
| 14 | 7 | $\pi / 2$ | [-1,0,7,0,-7,0,1]/[0,-4, $0,8,0,-4,0]$ | $\tan ^{4}(\Delta / 2)$ | 5 |
| 15 | 7 | $\pi / 2$ | [0,-2,0,4,0,-2,0]/[1,0,-3,0,3,0,-1] | $-\tan ^{2}(\Delta / 2)$ | 5 |
| 16 | 8 | $\pi / 2$ | [0,2,0,-4,0,3,0,-1]/[1,0,-3, $0,4,0,-2,0]$ | $\left[\frac{2 \cos (2 \Delta)-2 \cos (\Delta)+1}{2 \cos (2 \Delta)+2 \cos (\Delta)+1}\right] \tan (\Delta / 2)$ | 5 |
| 17 | 8 | $\pi / 2$ | [0,4,0,-11,0,8,0,-1]/[1,0,-8,0,11,0,-4,0] | $\left[\frac{2 \cos (2 \Delta)-1}{2 \cos (2 \Delta)+1}\right] \tan (\Delta / 2)$ | 5 |
| 18 | 9 | $\pi / 4$ | [0,1,2,1,0,-1,-2,-1,0]/[-1,-1,0,1,2,1,0,-1,-1] | $\frac{\cos (\pi / 8+\Delta / 2)-\sin (3 \pi / 8+3 \Delta / 2)}{\cos (\pi / 8+\Delta / 2)+\sin (3 \pi / 8+3 \Delta / 2)}$ | 5 |
| 19 | 1 | $\pi / 2$ | [1,0,-8,0,15,0,-15,0,8,0,-1]/[0,4,0,-12,16,0,-12,0,4,0] | $\tan ^{4}(\Delta / 2)$ | 5 |

## Acknowledgments

This work was partially supported by CONACyT under grant No. 42771.
\#113013 - \$15.00 USD Received 17 Jun 2009; revised 19 Jul 2009; accepted 21 Jul 2009; published 20 Aug 2009
(C) 2009 OSA

31 August 2009 / Vol. 17, No. 18 / OPTICS EXPRESS 15771

