

Phasorial analysis of detuning error in temporal phase shifting algorithms

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Abstract: Phase error analysis in Temporal Phase Shifting (TPS) algorithms due to frequency detuning has been to date only performed numerically. In this paper, we show an exact analytical expression to obtain this phase error due to detuning using the spectral TPS response. The new proposed method is based on the phasorial representation of the output of the TPS quadrature filter. Doing this, the detuning problem is reduced to a ratio of two symmetrical spectral responses of the quadrature filter at the detuned frequency. Finally, some popular cases of TPS algorithms are analyzed to show the usefulness of the proposed method.

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References and links

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1. Introduction

Among automated interferogram analysis methods, temporal phase shifting (TPS) techniques are considered as the most accurate wave-front extraction [1-8]. The measurement accuracy of the TPS depends on the estimated phase of the complex analytical signal associated with the real cosine signal given by the observed interferogram. The phase-shifting techniques utilize several TPS fringe patterns to determine this complex analytical signal whose angle is the desired signal to be estimated. All TPS algorithms give the exact fringe pattern phase (assuming very low measuring noise) when the temporal signal's carrier ω_0 has exactly the carrier frequency used in the TPS algorithm. When this is not the case, the actual temporal carrier used to obtain our interferograms differs from ω_0 and an erroneous phase is estimated. This erroneous phase estimation is called detuning error. To evaluate the detuning error from any quadrature filter used in TPS the common procedure has been numeric evaluation, then, it would be good to obtain an exact-analytical expression for the detuning to better understand

this phase estimation error [1-8]. The analytical expression herein derived for the detuning error in TPS algorithms allows us to minimize some TPS quadrature filters for detuning error. Most TPS algorithms used nowadays have no free parameters that permit us this kind of detuning optimization. However, some new techniques may be used to generate more general TPS algorithms including free parameters that allow us to improve the robustness of these new TPS filters to detuning. Even if no optimization is possible, one may always compare among the myriad of TPS algorithms available in the literature using the analytical expression given in this paper. The input of any TPS quadrature filter is a set of real cosine signals which are the intensity of a series of TPS interferograms captured by a CCD camera. The output of these TPS quadrature filters is the complex analytical signal at the temporal carrier frequency $-\omega_0$ associated with the real intensity of our interferograms. As mentioned before, when the carrier frequency of the interferograms is detuned, an additional undesired complex signal with carrier frequency $+\omega_0$ is added to our desired signal at $-\omega_0$. Hence the complex-vectorial (or phasor) sum of both signals give rise to an erroneously (detuned) estimated phase from this set of temporal interferograms.

2. Error Detuning in Phase-Shifting Interferometry (PSI)

The measured intensity of an interferogram on a CCD detector can be expressed as [3,4]:

$$\mathbf{I}(x, y, t) = a(x, y) + b(x, y) \cos[\phi(x, y) + \omega_0 t]. \quad (1)$$

Where, $\phi(x, y)$ denotes the unknown phase, $a(x, y)$ is the background illumination, and $b(x, y)$ is the contrast of interference fringes; these two signals are low frequency. The temporal carrier ω_0 is a linear phase shift among the set of interferograms which is introduced in the data gathering process. Meanwhile, t corresponds with the temporal sampling which is taken as a natural number in this paper. Taking the Fourier transform of $\mathbf{I}(x, y, t)$, we have $I(x, y, \omega)$ as,

$$I(x, y, \omega) = a(x, y) \delta(\omega) + \frac{b(x, y)}{2} \exp[-i\phi(x, y)] \delta(\omega - \omega_0) + \frac{b(x, y)}{2} \exp[i\phi(x, y)] \delta(\omega + \omega_0). \quad (2)$$

The output of the TPS quadrature filter being denoted by $g(t)$ is obtained by convolving a discrete temporal quadrature filter with several temporal phase shifted interferograms expressed as, $g(t) = h(t) * I(x, y, t)$ where (*) denotes one dimensional temporal convolution.

Taking the Fourier transform of this convolution product one obtains $G(\omega) = H(\omega)I(\omega)$. Notice that, the quadrature filter $h(t)$ is a one-sided (complex) convolution filter that is tuned at frequency ($\omega = \omega_0$). Quadrature filters are complex linear systems having a symmetric real component $hr(n)$ and an antisymmetric imaginary component $hi(n)$ or $h(t) = hr(t) + i hi(t)$. As a consequence, we have the very important fact that the resulting Fourier transform of $h(t)$ being $H(\omega)$ always becomes a real function of the frequency ω . Then, the output signal $G(\omega) = I(\omega)H(\omega)$ is expressed as,

$$G(\omega) = a H(\omega) \delta(\omega) + \frac{b}{2} H(\omega) \exp(-i\phi) \delta(\omega - \omega_0) + \frac{b}{2} H(\omega) \exp(i\phi) \delta(\omega + \omega_0). \quad (3)$$

where the spatial dependence (x, y) of the functions I , a , b and ϕ has been dropped. Now, we can observe in Fig. 1, that at (exactly) the carrier frequency $\omega = \omega_0$, we have $H(\omega_0) = 0$, $H(0) = 0$ and $H(-\omega_0) \neq 0$.

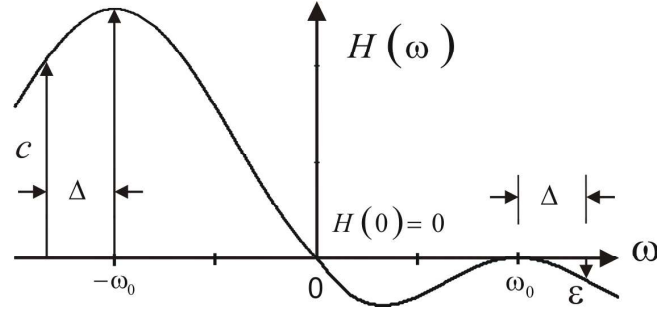


Fig. 1. Graphical representation of the detuning components c and ε .

Therefore, the spectrum of the analytical signal at the output of the TPS algorithm $G(\omega_0)$ becomes $G(x, y, \omega) = (b/2)H(-\omega_0)\exp(i\phi)$. Then the searched phase ϕ is recovered from the complex analytical signal $g(x, y, t)$ which is its inverse Fourier transform. Now let us consider that our phase stepping interferograms are not sampled at the expected frequency rate ω_0 , but at an erroneous temporal frequency given by $\omega = \omega_0 + \Delta$, then we have that the spectral response $G(x, y, \omega)$ becomes,

$$G(x, y, \omega) = \frac{b}{2}\exp(i\phi)H(-\omega_0 - \Delta)\delta(\omega + \omega_0 + \Delta) + \frac{b}{2}\exp(-i\phi)H(\omega_0 + \Delta)\delta(\omega - \omega_0 - \Delta). \quad (5)$$

The two (complex) output signals given by Eq. (5) are also graphically depicted in Fig. 1 in the frequency domain. We may also use a graphical-phasor representation of these two complex signals given in Eq. (5). This is shown in Fig. 2, where we have denoted the desired analytical signal as c and the undesired or spurious signal as ε ,

$$c = (b/2)H(-\omega_0 - \Delta), \quad \varepsilon = (b/2)H(+\omega_0 + \Delta). \quad (6)$$

then Eq. (5) becomes,

$$G(x, y, \omega) = c \exp(i\phi)\delta(\omega + \omega_0 + \Delta) + \varepsilon \exp(-i\phi)\delta(\omega - \omega_0 - \Delta). \quad (7)$$

Equation (7) may be graphically represented as phasors as in Fig. 2, where we show the desired signal c with an angle $\phi(x, y)$ and the erroneous spurious signal ε with angle $-\phi(x, y)$.

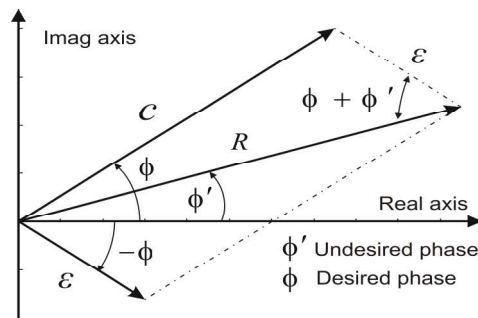


Fig. 2. Phasor representation of $G(x, y, \omega)$.

Now we proceed to the analysis of the phasor diagram given in Fig. 2, by applying the sinus law to the triangle formed by c , ε and R which stand for,

$$\frac{H(\omega_0 + \Delta)}{H(-\omega_0 - \Delta)} = \frac{\varepsilon}{c} = \frac{\sin(\phi - \phi')}{\sin(\phi + \phi')} = \frac{\tan(\phi) - \tan(\phi')}{\tan(\phi) + \tan(\phi')}. \quad (8)$$

The detuning error $\Delta\phi = \phi' - \phi$ may be defined as the difference between the desired phase ϕ and the undesired phase ϕ' . This expression for the detuning error $\Delta\phi$ is widely used to evaluate the robustness to detuning in TPS and it has been evaluated *only* numerically [1-8]. From Eq. (8) we obtain ϕ' , then by using this into $\Delta\phi = \phi' - \phi$, we find that the detuning error is expressed as,

$$\Delta\phi = \tan^{-1} \left[\left(\frac{c - \varepsilon}{c + \varepsilon} \right) \tan(\phi) \right] - \phi. \quad (9)$$

We can see in this expression that when $\varepsilon \rightarrow 0$ no detuning error is present and the erroneous phase ϕ' becomes the desired phase $\phi' \rightarrow \phi$. The main objective in this paper is to find an easy, useful and analytical detuning robustness equation in terms of the frequency response $H(\omega)$ of any TPS algorithm. To this end, we must substitute Eq. (8) into Eq. (9) and after using some trigonometric relations, the searched expression for the detuning error $\Delta\phi$ in terms of the frequency response of the TPS algorithm is,

$$\tan(\Delta\phi) = -(\varepsilon/c) \frac{\sin(2\phi)}{1 + (\varepsilon/c)\cos(2\phi)}. \quad (10)$$

This analytical expression is a novel, very simple and an exact way to evaluate the detuning error. This expression may be further simplified by taking into account that for all practical purposes $(\varepsilon/c) \ll 1.0$, as a consequence of this $\tan(\Delta\phi) \approx \Delta\phi$ the finally Eq. (10) becomes,

$$\Delta\phi \approx -\frac{\varepsilon}{c} \sin(2\phi) = -\frac{H(\omega_0 + \Delta)}{H(-\omega_0 - \Delta)} \sin(2\phi). \quad (11)$$

This shows the well known fact (without proof [1-8]) that the detuned phase always has a component which depends on the interferogram's fringes having twice the interference fringes which is $\sin(2\phi)$. The Eq. (11) is the most important result in this paper. A short-cut way to obtain the last result given by Eq. (11) from Eq. (8), is to consider that the detuning error is small. Therefore the following approximations apply $\phi \approx \phi'$, $\sin(\phi - \phi') \approx -\Delta\phi$ and $\sin(\phi' + \phi) \approx \sin(2\phi)$, as a consequence the detuning error becomes $\Delta\phi = -(\varepsilon/c)\sin(2\phi)$ which is equal to Eq. (11).

In the literature the maximum detuning error is found numerically [1-8]. To compare against these results, we must maximize our exact result in Eq. (10) with respect to ϕ obtaining the following analytical result,

$$\Delta\phi_{\max} = \sin^{-1} \left| \frac{H(\omega_0 + \Delta)}{H(-\omega_0 - \Delta)} \right| \approx \left| \frac{H(\omega_0 + \Delta)}{H(-\omega_0 - \Delta)} \right|. \quad (12)$$

We must emphasize that this result Eq. (12) for the maximum detuning error is exact (left side) so, it coincides exactly with the detuning error evaluated numerically elsewhere [1-8]. However, for small detuning a very good approximation is obtained and this is shown at the left hand side of Eq. (12). It should be notice that Eq. (12) always is valid for any $H(\omega)$ real or complex.

3. Some examples of error detuning in Phase-Shifting Interferometry.

In this section we analyze some popular TPS algorithms just for illustrative purposes to show how to apply the general techniques developed so far. Let us start with the Schwider-Hariharan five step algorithm [6],

3.1 The five frames algorithm:

The most used five steps TPS algorithm (Schwider-Hariharan) is given by [1,3,4,6]

$$\phi(x, y, \alpha) = \tan^{-1} \left\{ \frac{2[I(\alpha) - I(-\alpha)]}{[I(-2\alpha) + I(2\alpha) - 2I(0)]} \right\}, \quad \alpha = \pi/2 \quad (13)$$

where $\phi(x, y, \alpha)$ is the interferogram's phase for phase step $\alpha = \pi/2$. We first obtain the time response of the quadrature filter from the TPS formula Eq. (13) with as,

$$h(t) = [\delta(t+2\alpha) + \delta(t-2\alpha) - 2\delta(0)] + i 2[\delta(t-\alpha) - \delta(t+\alpha)], \quad \alpha = \pi/2 \quad (14)$$

where $i = \sqrt{-1}$; then, we obtain the frequency response $H(\omega, \alpha = \pi/2)$ of this quadrature filter as,

$$H(\omega, \alpha = \pi/2) = 4 \sin(\omega\pi/2) - 2[1 - \cos(\omega\pi)]. \quad (15)$$

This filter is tuned at frequency $\omega = 1$. Now assuming a detuning error in the phase step of $\alpha = \pi/2 + \Delta$ and according to Eq. (8) we obtain,

$$\frac{\varepsilon}{c} = \frac{H(\omega = 1, \alpha = \pi/2 + \Delta)}{H(\omega = 1, \alpha = -\pi/2 - \Delta)} = -\tan^2\left(\frac{\Delta}{2}\right). \quad (16)$$

Finally using this result into Eq. (10) we have the exact detuned phase error for the five frames TPS algorithm as,

$$\Delta\phi = \tan^{-1} \left[\frac{\tan^2(\Delta/2) \sin(2\phi)}{1 + \tan^2(\Delta/2) \cos(2\phi)} \right]. \quad (17)$$

This equation is the exact detuning error for any detuned step $\alpha = \pi/2 + \Delta$. This result may be approximated for small Δ (as reported in [3,4,6]) to obtain, $\Delta\phi = (\Delta^2/4) \sin(2\phi)$. It can be seen that the maximum detuning is when the sine function equals one so $\Delta\phi_{\max} = \Delta^2/4$ renders the well known quadratic detuning error behavior of the five step Schwider-Hariharan TPS algorithm [5,7].

3.2 Seven frames algorithm

The seven steps TPS algorithm given by the Servin's algorithm [8] is,

$$\tan[\phi(x, y, \alpha)] = \frac{I(-3\alpha) + 4.3I(-2\alpha) - 14I(-\alpha) + 14I(\alpha) - 4.3I(2\alpha) - I(3\alpha)}{1.5I(-3\alpha) - 6I(-2\alpha) - 4.5I(-\alpha) + 18I(0) - 4.5I(\alpha) - 6I(2\alpha) + 1.5I(3\alpha)}. \quad (18)$$

Then, the frequency response of this TPS algorithm is,

$$H(\omega) = 2[\sin(3\omega\alpha) + 1.5 \cos(3\omega\alpha) + 4.3 \sin(2\omega\alpha) - 6 \cos(2\omega\alpha) - 14 \sin(\omega\alpha) - 4.5 \cos(\omega\alpha) + 9] \quad (19)$$

This quadrature filter is tuned at $\omega = 1$ for a phase step $\alpha = \pi/2$, and we have that, $H(1) = 0$ and $H(0) = 0$. Now let us detune our phase step to $\alpha = \pi/2 + \Delta$ then we obtain,

$$\Delta\phi_{\max} = \sin^{-1} \left| \frac{\cos(3\Delta) + 4.3 \sin(2\Delta) + 14 \cos(\Delta) - 1.5 \sin(3\Delta) - 6 \cos(2\Delta) - 4.5 \sin(\Delta) + 9}{\cos(3\Delta) + 4.3 \sin(2\Delta) + 14 \cos(\Delta) + 1.5 \sin(3\Delta) + 6 \cos(2\Delta) + 4.5 \sin(\Delta) + 9} \right|. \quad (20)$$

Using the first order approximation given by $\cos(n\Delta)=1.0$ and $\sin(n\Delta)=n\Delta$ for any n in Eq. (20) the maximum detuning error is

$$\Delta\phi_{\max} = \left| \frac{\Delta}{75 + 44 \Delta} \right|. \quad (21)$$

We can observe the extremely low sensitivity to the detuning reported *only* numerically in [4,8].

3.3 Eleven frames algorithm

Finally, just for illustrative purposes we briefly analyze the eleven steps TPS algorithm (De Groot in [3]) which is,

$$\phi(x, y, \alpha = \pi/2) = \tan^{-1} \left\{ \frac{[I(5\alpha) - I(5\alpha)] - 8[I(-3\alpha) - I(3\alpha)] + 15[I(-\alpha) - I(\alpha)]}{4[I(-4\alpha) + I(4\alpha)] - 12[I(-2\alpha) + I(2\alpha)] + 16I(0)} \right\}. \quad (22)$$

Applying the same easy procedure as before we obtain,

$$\Delta\phi_{\max} = \sin^{-1} \left| \tan^4(\Delta/2) \right|. \quad (23)$$

For small Δ we may approximate this detuning error as $\Delta\phi_{\max} \approx \Delta^4/16$.

As a final point, we graphically compare in Fig. 3 all the TPS algorithms that were analyzed.

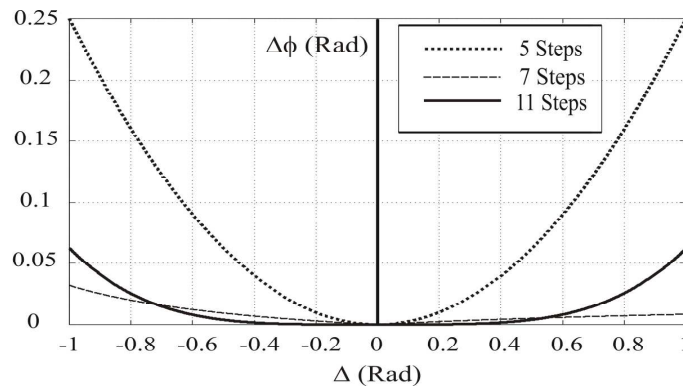


Fig. 3. Error detuning for TPS algorithms for five, seven and eleven steps.

4. Conclusions

An exact, analytical algorithm to evaluate the error detuning in phase shifting algorithms was obtained from a phasor representation of the detuned output signal of any TPS algorithm. The derived expression was applied to evaluate useful approximations to the error detuning from some well known quadrature filters with five, seven, and eleven steps.

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