Non-iterative method for designing super-resolving pupil filters

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Abstract: We propose a method of designing pupil filters for transverse super-resolution without making use of recursive algorithms or the parabolic approximation for the point spread function (PSF). We represent the amplitude of the PSF as an expansion of orthogonal functions from the Fourier-Bessel transform of a Dini series. Their coefficients are related with desired features of the PSF, such as the transversal super-resolution gain and the intensity of the secondary maxima. We show the possibility to derive closed formulas to obtain large super-resolution gains with tolerable side-lobe intensities, at the expense of increasing the intensity of a chosen secondary lobe.

OCIS codes: (110.0180) Microscopy; (110.1220) Apertures; (170.1790) Confocal Microscopy.

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1. Introduction

There are fields where it is crucial to improve the resolution limit of an optical system. For example, in laser ablation the PSF diameter needs to be reduced for femtosecond laser machining [1]; in microscopy, such improvement is required in confocal microscopes [2]; in high density data storage, super-resolving confocal readout systems are needed to retrieve the

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data [3], etc. In general, super-resolution involves a transversal or a longitudinal reduction of the PSF, although these are not independent [4]. In particular, the design of super-resolving pupils demands the reduction of the PSF radius, the optimization of the Strehl ratio and the brightness of its rings. Toraldo Di Francia [5] proposed an interpolating method to design pupils with equally spaced rings, and proved the possibility of reducing the size of the PSF as much as desired by displacing the brighter secondary lobes farther from the central lobe. The cost was a drastic reduction of the Strehl ratio. Later, Sheppard and Hegedus [6] introduced the parabolic approximation of the PSF, which has been used to design pupils with iterative methods [7–9], or to derive closed expressions in the case of pupils with three zones [10]. More recently, the use of parabolic approximations has been abandoned, and instead accurate merit functions have been constructed to use in conjunction with iterative methods [11–13]. In any case, these methods require an initial guess, which in most cases determines the solution that we obtain.

The aim of this work is to show the possibility of designing useful super-resolving pupil filters, without resorting to iterative schemes and second order approximations for the PSF (parabolic approximation). The desired pupil function is expanded into a Dini series, and its Fourier-Bessel transform is found analytically. This also leads to a series in terms of a set of orthogonal functions, but now representing the amplitude of the PSF. Previously, one of the authors used this kind of series to find super-resolving pupils with iterative methods [14]. What we want now is to avoid the iterative scheme, by relating the coefficient of the amplitude PSF series to parameters such as the super-resolving gain and the maximum intensity of the secondary lobes. In this form, we can readily obtain new super-resolving pupils by the simple expedient of providing the coefficients of the Dini series, which are found from simple algebraic expressions involving the parameters just mentioned above. Although the method becomes less accurate outside the interval $0.55 < \varepsilon < 0.75$, where $0 < \varepsilon < 1$ is the super-resolving gain parameter, we can still achieve with relative ease large reductions of the central lobe of the PSF, with tolerable side-lobe intensity ratios. For this we have to accept a drastic drop of the Strehl ratio, arising from channeling a large portion of the energy in the PSF to a predefined secondary lobe. This resembles, of course, the result obtained by Toraldo di Francia with binary pupils [5]. In the last part of the paper we present some examples that clearly show this kind of results.

2. Methodology

We shall start with a derivation of the expansion for the PSF, and its connection with the pupil function. Let $G(\delta, v)$ be the normalized complex amplitude distribution of an axially symmetric, complex pupil function $g(\rho)$. Then

$$G(\delta, v) = 2\int_0^1 g(\rho)\exp(-i2\pi\delta\rho^2)\,J_0(v\rho)\,\rho\,d\rho\,, \tag{1}$$

where ρ is the normalized radial coordinate and (δ, v) are the axial and the transverse dimensionless optical coordinates, respectively, defined by the equations $\delta = \xi(NA)^2/2\lambda$ and $v = 2\pi r\,NA/\lambda$, where $\xi = z - f$ is the axial distance from the focus, *NA* is the numerical aperture, *r* is the radial distance and λ the wavelength. Let us now suppose that we can describe the pupil function as a truncated Dini series:

$$g(\rho) = \sum_{n=0}^{K} \frac{C_n}{J_0^2(\alpha_n)}\,J_0(\alpha_n\rho)\,, \tag{2}$$

where $J_0(x)$ is the Bessel function of the first kind and zero order, α_n are the roots of $J_1(x)$, C_n are the, possibly complex, coefficients to be calculated, and $K + 1$ the number of basis

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functions that we adopt. We must recall here that the pupil function must satisfy the condition $|g(\rho)| \le 1$ in the interval [0, 1].

To obtain the corresponding transverse amplitude of the PSF we use Eq. (2) in Eq. (1) and set $\delta = 0$. We obtain:

$$\phi(v) = G(0,v) = \frac{1}{J_0^2(\alpha_n)} \sum_{n=0}^{K} C_n \; 2\int_0^1 J_0(\alpha_n \rho) J_0(v\rho)\rho \, d\rho . \tag{3}$$

Evaluating the integrals as shown in [15], we can rewrite Eq. (3) as

$$\phi(v) = \sum_{n=0}^{K} C_n \, \phi_n(v) , \tag{4}$$

where

$$\phi_n(v) = \frac{1}{J_0(\alpha_n)} \frac{2v J_1(v)}{v^2 - \alpha_n^2} . \tag{5}$$

From this equation it can be readily shown that $\phi_n(\alpha_k) = \delta_{nk}$, where δ_{nk} is the kronecker delta; in other words, every basis function $\phi_n(v)$ nulls at the location of the maxima of the others [15]. Using this property in Eq. (4) we find

$$\phi(\alpha_k) = C_k . \tag{6}$$

Therefore, the coefficients that we seek are the amplitudes of $\phi(v)$ evaluated at the roots of $J_1(x)$. In general the location of the peaks of the PSF, $|\phi(v)|^2$, it is not at the roots α_n, but it is close to them if $0.55 < \varepsilon < 0.75$, given that the basis functions $\phi_n(v)$ have their principal maximum precisely at α_n [15].

We shall now proceed to adjust the shape of the function $|\phi(v)|^2$, taking into account certain parameters. These are the super-resolving gain ε, the Strehl ratio *S*, and the relative intensity of the first lobe Γ_1 [12]. They are defined as follows:

$$\varepsilon = \frac{D}{D_c} , \tag{7}$$

$$S = \frac{|\phi(0)|^2}{|\phi_c(0)|^2} , \tag{8}$$

$$\Gamma_n = \frac{|\phi(0)|^2}{|\phi(\mu_n)|^2} , \qquad n = 0,...,K, \tag{9}$$

where *D* is the diameter of the central lobe of the PSF, the sub index *c* stands for clear aperture, and μ_n is the "optical" radius of its *n*th-secondary maximum. Note that we have extended the definition of the parameter Γ_1 to all side-lobe intensities.

We will now relate the coefficients of Eq. (4) with these parameters. Since $\mu_n \simeq \alpha_n$, recalling that $\alpha_0 = 0$, from Eqs. (6) and (9) we obtain:

$$\frac{|C_0|}{|C_n|} = \frac{|\phi(\alpha_0)|}{|\phi(\alpha_n)|} \simeq \frac{|\phi(0)|}{|\phi(\mu_n)|} = \sqrt{\Gamma_n} , \qquad n = 0,...,K , \tag{10}$$

with $\Gamma_0 = 1$. Assuming $\left|\phi_c(0)\right|^2 = 1$, from Eq. (6) and (8) we have $\left|C_0\right| = \left|\phi(0)\right| = \sqrt{S}$. Therefore

$$\left|C_n\right| \simeq \sqrt{\frac{S}{\Gamma_n}}, \qquad n = 0, ..., K. \tag{11}$$

Since Eq. (11) only involves the modulus of the coefficients, in particular we can choose

$$C_n \simeq \sqrt{\frac{S}{\Gamma_n}} \exp(in\pi) = (-1)^n \sqrt{\frac{S}{\Gamma_n}}, \qquad n = 0, ..., K. \tag{12}$$

Once we have related the values of Γ_n with the coefficients of the expansion, we turn our attention to the super-resolving gain parameter ε, pertaining to the reduction of the central lobe of the PSF. What we want is [Eq. (4)]:

$$\phi(\varepsilon\alpha_1) = \sum_{n=0}^{K} C_n \phi_n(\varepsilon\alpha_1) = 0, \qquad \varepsilon < 1. \tag{13}$$

In order to satisfy Eqs. (13) and (12), one of the coefficients in Eq. (13) must be calculated from the rest. For example, the last one:

$$C_K = \frac{\sqrt{S}}{\phi_K(\varepsilon\alpha_1)} \sum_{n=0}^{K-1} \frac{(-1)^n}{\sqrt{\Gamma_n}} \phi_n(\varepsilon\alpha_1). \tag{14}$$

With this, we also fix the value of Γ_K [Eqs. (14) and (12)]. At this point we have obtained the coefficients C_n that, substituted in Eq. (4), will give us the PSF with the desired modifications, namely, a reduced central lobe, and known side-lobe relative intensities. However, these coefficients must be scaled to ensure that the condition $\left|g(\rho)\right| \le 1$ will be satisfied in the interval [0, 1]. The new coefficients, therefore, will be $k\,C_n$, with $k = 1\big/\max\left[\left|g(\rho)\right|\right]$, where $\max\left[\left|g(\rho)\right|\right]$ is computed from Eq. (2) and the original coefficients; typically we obtain $k < 1$. This normalization procedure does not affect the relative intensities of the secondary lobes, which are given by ratios of the coefficients, nor the value of the super-resolution gain. But the actual Strehl ratio becomes $k^2\left|C_0\right|^2 = k^2 S$, where S was its intended value. Thus, with our method the parameters ε, S and Γ_1 of Eqs. (7), (8) and (9) cannot be satisfied independently, since the value of S must be used to ensure that $\left|g(\rho)\right| \le 1$ in the interval [0, 1] [Eqs. (12), (14) and (2)]. For simplicity, therefore, we will set $S = 1$ to start the design of any pupil filter.

3. Results and discussion

We shall now present some examples to clarify further our method of design, and also to compare some of our results with those obtained in previous work. In the first example we used two coefficients ($K = 1$) and $\varepsilon = 0.60,\ 0.63$ and 0.65, to show that two basis functions are sufficient to design pupil filters with moderate Strehl ratios and tolerable side-lobe intensities. Initially we set $C_0 = \sqrt{S} = 1$ [Eq. (12)], and calculate C_1 for each value of ε [Eq. (14)]. Substituting the values of these two coefficients in Eq. (2) we obtain the normalizing factors k, and from them the correct values for S. For example, in the case of $\varepsilon = 0.65$ we had $S = 0.174$, in agreement with the corresponding plot in Fig. 1. From the heights of the central and the first peaks in this plot we find $\Gamma_1 \simeq 2.6$. The pupil functions corresponding to the PSFs shown in Fig. 1 can now be obtained from Eq. (2). Figure 2 shows the pupil function for the case $\varepsilon = 0.65$. For the sake of comparison, we also included the case $\varepsilon = 0.80$.

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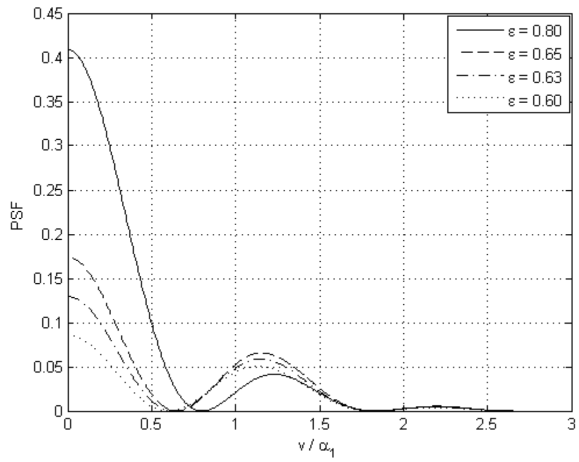

Fig. 1. PSFs obtained with our method for different values of the super-resolution gain parameter: $\varepsilon = 0.80,\ 0.65,\ 0.63,\ 0.60$. Only two basis functions (K = 1) were employed in this example.

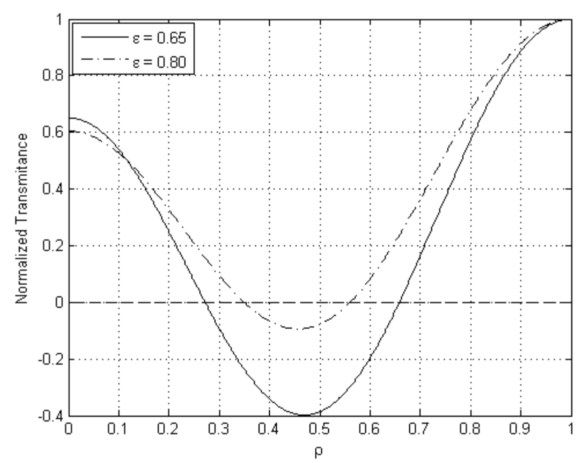

Fig. 2. Pupil functions corresponding to the PSFs with $\varepsilon = 0.65$ and 0.80 in Fig. 1.

In the second example we choose K = 2, to show how can we manipulate the intensity of one of the side lobes. Using again $\varepsilon = 0.65$, we assigned different values to Γ_1, obtaining in this form PSFs with different values of S and $C_{K=2}$ - and thus of Γ_2. For example, setting $\Gamma_1 = 3.0,\ 3.5$ and 4.5, after scaling the coefficients in each case we found that S = 0.131, 0.155, 0.100 (Fig. 3). Note that a higher value of Γ_1 does not necessarily mean a lower value of S. In particular, the three coefficients which yielded S = 0.100 were (0.3147, –0.1484, 0.0951). Substituting them in Eq. (4) and plotting the resulting PSF we can see that in fact $\varepsilon = 0.65$, S = 0.100 but $\Gamma_1 = 4.2$ (curve with a dotted line). The value that we expected for Γ_1 was $C_0^2 / C_1^2 = \left[0.3147/(-0.1484)\right]^2 \simeq$ 4.5. As we mentioned before, the discrepancy between the expected and the actual value of Γ_1 arises from the fact that we assumed that Γ_n could be obtained from the values of $\left|\phi(\alpha_n)\right|^2$, instead of those of $\left|\phi(\mu_n)\right|^2$, where μ_n is the location of the *n*th maximum of the PSF. In practice, the tails of the basis functions adjacent to the *n*th basis function add small contributions to the PSF in the vicinity of α_n, shifting the position of the local maximum of the PSF from α_n, the position of the maximum of the *n*th basis function, to μ_n, the actual location of the *n*th maximum of the PSF. We believe that this is not a serious drawback, however, giving that the desired value of Γ_n may be obtained by increasing slightly its intended value. There is no such problem with the super-resolving gain parameter ε, which is predicted accurately. Figure 4 shows the relative errors of Γ_1 for various nominal values of this parameter. In all cases $\varepsilon = 0.65$ and $K = 2$. Notice that the percentual error is under 20% in the range $0.55 < \varepsilon < 0.75$, for all the values of Γ_1 that we chose.

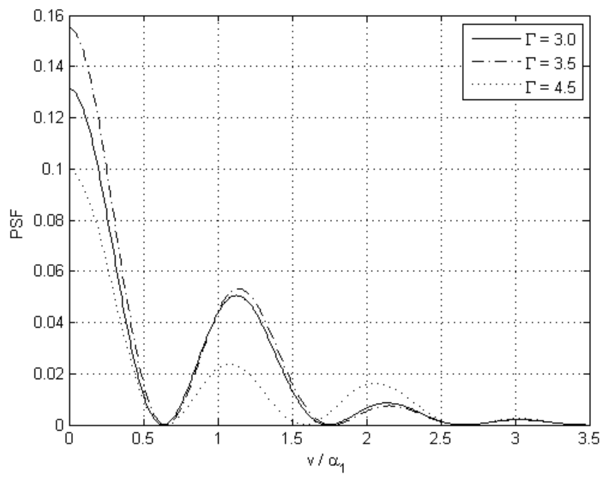

Fig. 3. PSFs obtained with our method for $\Gamma_1 = 3.0,\ 3.5,\ 4.5$. In all cases $K = 2$ and $\varepsilon = 0.65$. Notice that when $\Gamma_1 \simeq 4.5$, $\Gamma_2 \simeq \Gamma_1$.

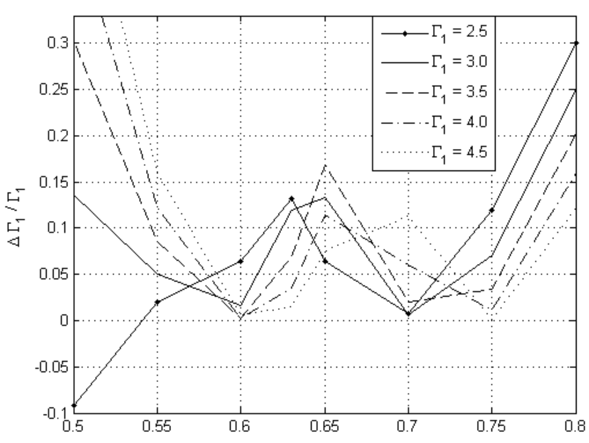

Fig. 4. Relative errors for various nominal values of the parameter Γ_1. In all cases $\varepsilon = 0.65$ and $K = 2$.

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The third example shows the possibility of further reductions of the super-resolution gain parameter, at the expense of a large increment in the intensity of a predefined side-lobe. Figure 5 shows a PSF with $\varepsilon = 0.50$, obtained with only four coefficients of the Dini expansion (K = 3). We set $\Gamma_1 = \Gamma_2 = 2.5$ and calculated C_3 from Eq. (14). After scaling the resulting coefficients we had $C_n = (0.0340, -0.0215, 0.0215, -0.0711)$. Plotting the corresponding PSF we can see that in fact $\varepsilon = 0.50$, $S = (0.0340)^2 = 0.0012$, but $\Gamma_1 \simeq 2.2$. The reason for the slight discrepancy in the value of Γ_1 was given above.

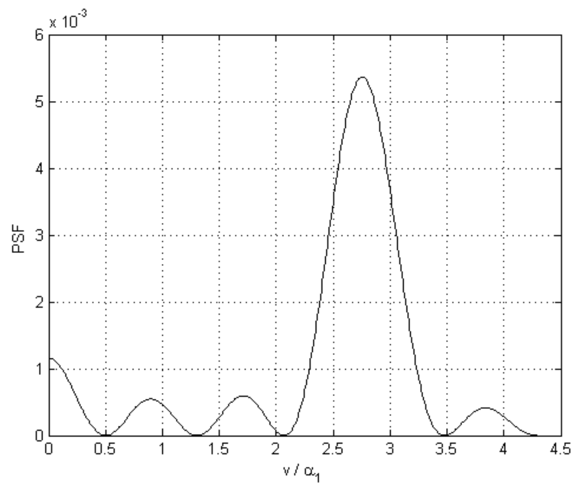

Fig. 5. PSF designed for a high super-resolving gain parameter, $\varepsilon = 0.50$, with $\Gamma_1 = \Gamma_2 = 2.5$.

Finally, in Fig. 6 we compare our results (the curve with a dotted line of Fig. 3), with those of Canales and Cagigal (the curve with an interrumped line in Fig. 6) [12]. Both results were normalized to show the behavior of the secondary lobes. Our PSF shows a higher reduction of the secondary lobes, but this was achieved with a lower value of the Strehl ratio, *S*. Similar results were obtained by one of the authors when using an iterative method for the same design problem [14].

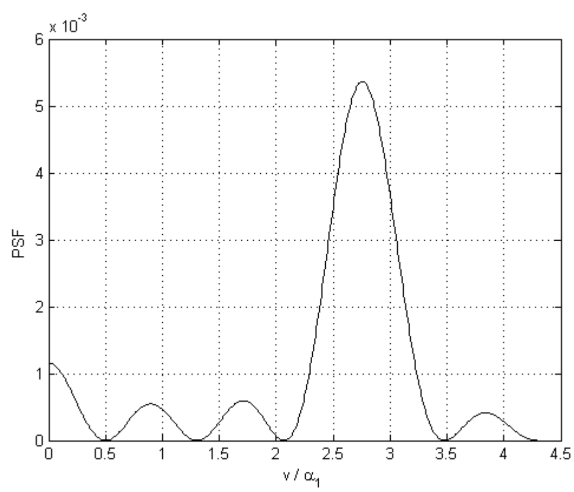

Fig. 6. Comparison of the PSF obtained with our method and the best PSF presented in ref [12]. For the purposes of comparison, both PSFs have been normalized.

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4. Conclusions

We have presented a non-iterative method to design super-resolving pupils based on Dini series, obtaining formulas which relate the coefficients of the series with relevant design parameters, like the super-resolution gain and the relative intensities of the side lobes of the PSF. Both, the pupil function and its corresponding PSF can be readily computed from these coefficients.

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