CENTRO DE INVESTIGACIONES
EN OPTICA, A.C.

"LOCAL CURVATURE CALCULATION AND ITS USE IN IMPROVED TOPOGRAPHIC MAPS."

Tesis que para obtener el grado de Doctor en Ciencias (Óptica)

Presenta: José Hernández Delgado

Director de Tesis: Dr. Daniel Malacara Hernández

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D. Malacara H.

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Summary

In this thesis, we develop the proper mathematical tools to calculate the local curvature over any type of surface including the cornea of a human eye. The calculus of local curvature of a curve described on a plane and the common methods in ophthalmology are used to calculate and graph the curvature or power maps. Then, we extend the calculus of local curvature to surfaces, and we propose a general equation to calculate the curvature over any hypothetical mathematical surface. We connect the general equation with cases of the calculus of curvature and we show how to obtain any case from the general curvature equation. We transform the general curvature equation to the Euler equation, this form is useful to describe the Cassini ovals, which can be used to represent in a geometric manner all the cases of curvature description used in the fields of ophthalmology, tribology, and in topography. We propose a new and interesting manner to graph the common representation of a wavefront utilizing the Cassini ovals over a circular pupil. Euler equation uses the principal curvatures and the angle to graph its polar form. This approach improves the curvature description and gives more information than those of the classical color-coded maps. Examples of Cassini maps are displayed to different wavefront representations to validate the advantages over color-coded maps.

Resumen

En el presente trabajo de tesis se desarrollaron las herramientas matemáticas para el cálculo de curvaturas locales sobre cualquier tipo de superficie incluido superficies sin simetría de rotación respecto a un eje óptico. Se incluye la córnea como ejemplo de superficie asimétrica. Por lo que el enfoque del trabajo de tesis se desarrolla en el ámbito de la oftalmología, en donde es de gran ayuda para el especialista el cálculo de mapas de curvatura o mapas de potencia dióptrica. Se extendió y se trató el cálculo de curvatura local sobre superficies con el propósito de calcular en cualquier curva o frente de onda hipotético. Se abordó la conexión de la ecuación general con los casos particulares y demostramos cómo obtener estos casos. Además, se utilizó la forma de la ecuación de Euler para describir mejor las superficies con poca simetría. La forma polar de la ecuación de Euler llamados óvalos de Cassini fueron utilizados para hacer una mejor representación de la dirección de la curvatura local. Se propuso una nueva metodología gráfica, así como un algoritmo para representar el comportamiento de la curvatura local en superficies hipotéticas de frente onda. La nueva propuesta para la representación de mapas de curvatura proporciona nueva información y mejora la representación clásica de mapas de color.

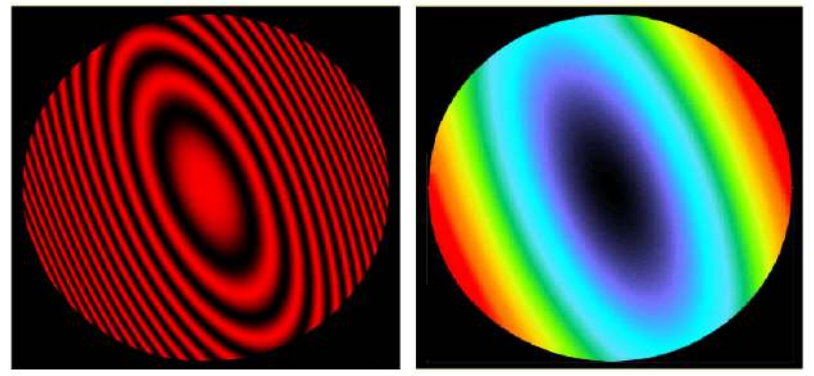
1 Introduction

In this work of thesis, we study the light as wavefront, this concept is very useful in optics and the ophthalmic field. Commonly the wavefront is calculated for optical systems in optical testing. From these optical tests, we can obtain the transverse aberration values and then fit the resulting in a wavefront model. These wavefront calculations are used as an evaluation of the optical system and in the case of ophthalmology to evaluate the corneal surface, which is responsible for 70% of the refractive power in the human eye [1].

Traditionally the calculus of curvatures in optics, considers that the surface (wavefront) has revolution symmetry and involves that some important characteristics of the surface are omitted, as is the case of high order aberrations and problems of vision in humans that develops precise jobs as fighter pilots, shooting athletes and problems with the night vision. These aberrations can be represented by Zernike polynomials and traditionally vision correction focuses on the correction of low-order aberration [1].

The equation of local curvature is related to the inverse of the radius of an osculating circle of radius r. This osculating circle is given by the best osculating circle according to the Euler equation to the normal curvature equation. Using these calculations of curvature values in the power equation we can draw and graph the color maps that are commonly used in ophthalmological optics. The dioptric power distribution of the cornea in the case of the human eye evaluation is used to diagnose vision related problems. We can assume that a healthy cornea has constant curvature at each point. However, the main problem is that due to aberrations, pathologies such as keratoconus and LASIK procedures the cornea surface is described into an irregular surface and color-coded maps are underestimated [2].

The main graphical tool to show wavefront aberrations is by color-coded 2D graphical representation as level, power, and curvature maps, other principal graphs are by fringes in the Twyman-Green interferograms [3], [4]. When evaluating most optical systems or surfaces, the wavefront or surface topography becomes extremely important. Typically, it is show in a topographic map obtained by optical interferometric methods, for example, the Twyman Green interferometer in Fig. 1(a), appear an astigmatic (sphero-cylindrical deformation) which is a topographic map with equal elevation deformation, where each line means the geometric locus of points of equal elevation. The difference in elevation between two consecutive bright fringes is one wavelength. Figure 1(b) is an equal elevation figure for the constant deviation points, corresponding to Fig. 1(a), where the different colors represent different values of the elevation.



(a) (b)

Figure 1. - Maps of equal elevation deformation in a wavefront or surface of an optical system. a) Twyman-Green interferograms and b) colored-coded map.

An advantage of the Twyman-Green interferogram is that it provides more quantitative results than the color-coded map, for example: shape and wavefront, but a disadvantage is that its elevation has a sign uncertainty that the color-coded map does not have. In the Twyman-Green interferograms, the fringe separation is an indication of the elevation slope.

Intuitively, we know that the local curvature tells us how fast the elevation slope changes at some point. In some optical systems, the local curvature instead of the elevation becomes more important since the value of the curvature is an indicator of the local convergence or divergence power of the optical surface or the local degree of convergence or divergence in a wavefront. This will be described in more detail later.

The geometrical concept of curvature and the mathematical theory of local curvatures are quite old, and they are the main subjects of the books about differential geometry. The history of the concept of curvature has been treated recently by many authors with some detail [5], [6]. The first attempts to formally define the curvature started many centuries ago, beginning with the Greek writers, following with Johannes Kepler (1571-1630) who was the first to define the curvature as the inverse of the local radius of curvature. However, the first successful study was that of Sir Isaac Newton in 1671, as described by Coolidge (1952) [7]. He said that the curvature of a curved line is equal to the curvature of the largest circle that is tangent to the curve on its concave side and that the center of the circle is the center of curvature. The study of curvatures is increasing its importance day by day and has recently become a very important concept in optometry and ophthalmology. It is thus surprising that classical optics books ignore this subject almost completely. Even ophthalmology and optometry books study curvatures at just an introductory level. Mathematicians and specialists in this field are the ones that study this subject in detail and great advances have recently been made. It is the purpose of this Thesis to make a general review of the subject at a level so that opticians, optical engineers, and in general non-specialists in the field can get an introductory description of the main concepts today in wide use. It is the purpose of this Thesis to make a general review of the subject at a level so that opticians, optical engineers, and in general non-specialist in the field can get an introductory description of the main concepts today in wide use.

When evaluating an optical surface, we might be interested in the map of the optical surface or wavefront deformations, frequently called aberrations, as given by an elevation map. A typical example includes the surface of a primary mirror for an astronomical telescope under polishing or figuring. In some other optical surfaces, the important characteristics to be measured are not the optical surface or the wavefront deviation but the local curvatures, as in the case of the human corneas. The most evaluated curvatures in optometry and ophthalmology are the sagittal and tangential curvatures and the axis orientation [8]. In most commercial instruments, this is done with approximations, frequently with the assumption that the optical surface or wavefront are nearly rotationally symmetric. More general and precise evaluation methods were not found to be described in any general optics, ophthalmology, or optometry books. Surprisingly, to our knowledge, this information was absent even in differential geometry books.

The subject of surface topography and shape, of surfaces, studied in differential geometry is quite important and useful in many fields, mainly in ophthalmic and optometric optics and in geology [9]–[12], [12]–[14]. Typically, in these books, the first fundamental and the second fundamental forms of the surfaces are developed, leading to expressions for the principal curvatures but not for the sagittal and tangential curvatures. In optics, these concepts are useful for the study of many properties of optical surfaces and wavefronts. The basic intrinsic properties of a surface are described by the fundamental forms of surfaces, which are the subject of differential geometry books. Their results can be applied to study the sagittal and tangential curvatures, but frequently some other methods are used [15].

In 1993 Steven E. Wilson published a study about the standardization of Color-Coded Maps for corneal topography [15]. In this study, they make a description about the differentiation in color-coded scale and provide a better understanding of the color maps using only a different scale in dioptric power. However, some characteristics of the surface are hidden.

2 Cornea

The cornea and crystalline lens are the two most important components of the human eye and the cornea consist of five layers: epithelium (1.401 to 1.433 refractive index), Bowman's membrane, stroma (1.357 to 1.38 refractive index), Descemet's membrane, and endothelium. The first 50 microns are related to the epithelium, Bowman's membrane has about 5 to10 microns, stroma 550 microns, Descemet's membrane is a thin acellular layer that is as a basement for endothelium that consists of a monolayer of cells as we see in Figure 2, and in this thesis, we use the mean refractive index of 1.3765.

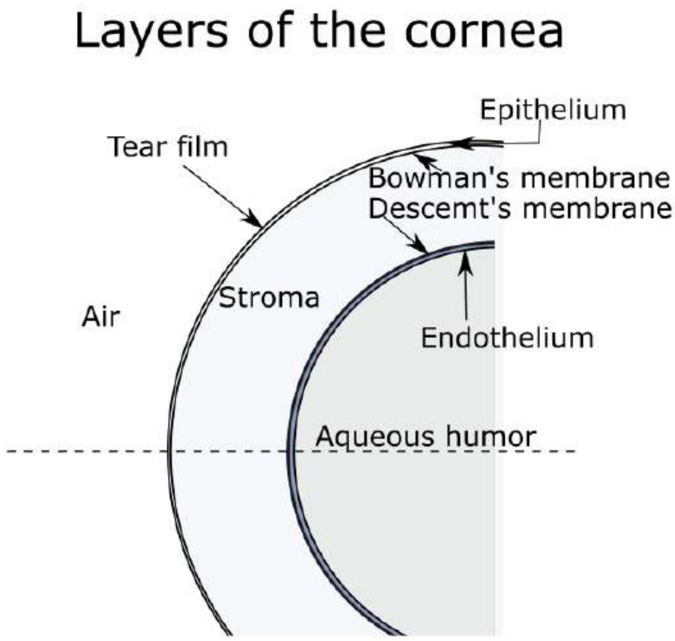


Figure 2. - Principal parts of the Structure of the Cornea

The most common description of the shape of the cornea is a prolate ellipse and the common radius used is about 7.7-7.9 mm [16]. The center of the cornea is difficult to define and is assumed to be the relative location of the center. The pupil and the pupil axis is along the line perpendicular to the cornea and passes through the center of the entrance pupil. In terms of the curvature the cornea exhibits a high degree of toricity [16].

2.1 Corneal Curvature

The power ***P*** of the surface of the cornea is calculated with the following equation:

$$P = \frac{(n-1)}{r} \tag{2.1}$$

Where *r* is the radius of curvature and *n* is the refractive index of the cornea. This formula is correct only for symmetric surfaces, for example to a sphere. In this thesis we derive a more accurate way to calculate parameter.

2.2 Refractive error

The human vision works ideally matching a fixation point on the fovea; however, refractive errors (myopia, astigmatism, etc.) changes that point out of the fovea. In Fig 3 we can observe how the axial length, δl, is modified by a refractive power error. A small change in the cornea surface produces an error in the dioptric power of the eye.

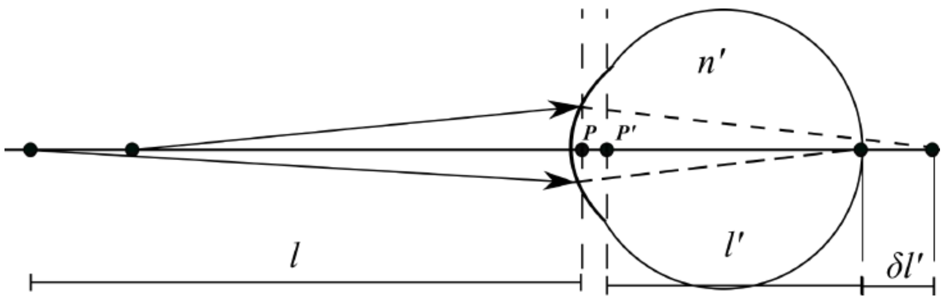


Figure 3. - Axial length error by a change in the refractive power.

This shape can affect the level of ametropic, using a schematic eye (Fig. 3) that error can be calculated. In the same way a change in the radius of curvature of the cornea leads to a change in corneal power.

Figure 4 shows a small error, $\delta l'$, where the n' is the refractive index, l and l' are the distances of work. The rays fall outside the retina and can be measured axially.

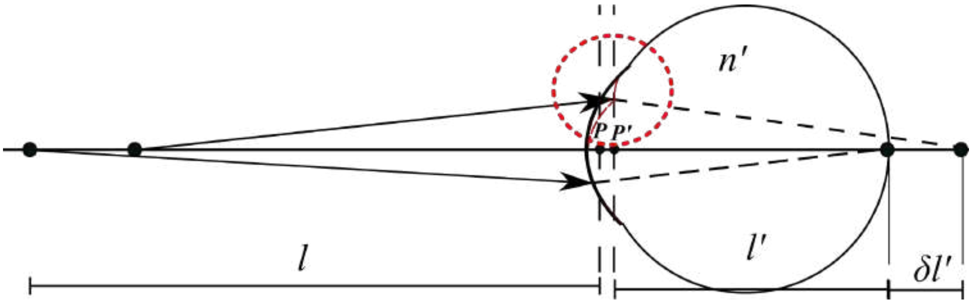


Figure 4. –Axial length error due to a change in curvature in the cornea.

Figure 5. Shows an example of a spherical wavefront over a plane (two dimensional space). Ideally, a point light source generates this spherical wavefront the location of the point source corresponds to the center of curvature of the spherical wavefront.

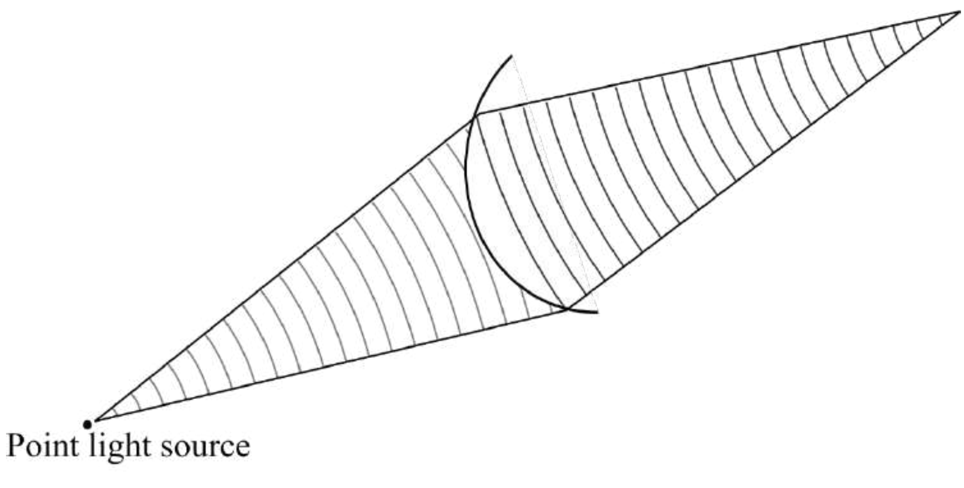


Figure 5. –Wavefront curvature due to a point light source and a point light image.

Curvature

In the Fig 6. We show an example where the incident wavefront is modified to compensate aberrations. This figure shows a Schmidt camera that has a correcting plate, with a varying curvature on the first surface, where the wavefront is changed to compensate for spherical aberration.

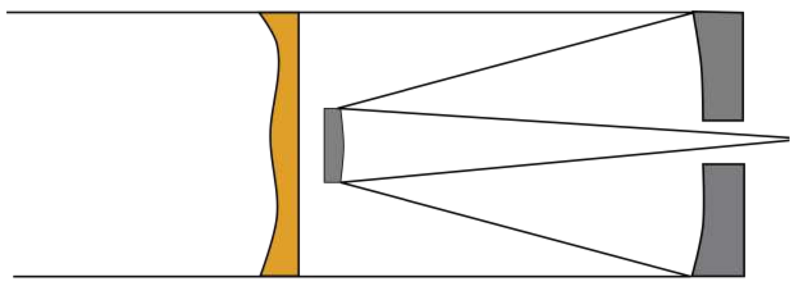


Figure 6. –Schmidt Cassegrain camera with variable curvature.

2.3 Zernike polynomials

A polynomial expression for describing optical surfaces properties is by Zernike polynomials.

In ophthalmic optics, the most used surfaces are the sphere and the cylinder. High order of Zernike polynomials is more related with some pathologies as keratoconus and after LASIK (laser-assisted in situ keratomileusis) procedures.

An optical surface shape can be described by the equation:

$$z(\rho,\,\theta)=\frac{c\rho^{2}}{1+\left[1-(K+1)c^{2}\rho^{2}\right]^{1/2}}+\sum_{j=1}^{4}A_{n}\rho^{2(n+1)}+\sum_{n=1}^{N}\sum_{m=1}^{M}B_{nm}Z_{n}^{m}(\rho,\theta) \tag{2.2}$$

Where the first term is a representation of a sphere, ellipsoid or a hyperboloid depending on the conic constant K (K = 0 spheres, 0 > K > -1 ellipses, K= -1 parabolic surfaces), $\rho^2 = x^2 + y^2$, c is the curvature of the reference surface that in most cases we consider to be a sphere, An are the deformation coefficients and the last term is a linear combination of Zernike Polynomials Z_n^m and coefficients B_n^m .

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For Zernike polynomials, term Zero is called piston and it represents a delay in the wavefront without degradation of the image quality. Primary aberrations are spherical, coma, astigmatism, the field of curvature and distortion [17].

3 Surface Theory:

When we use the concept of surface, in optics, we refer to a wavefront that comes from an optical system. However, in this chapter, surfaces are considered in the way of Differential geometry. A regular surface is defined in terms of three-dimensional Euclidean space coordinates that depend on the real parameters "*x*" and "*y*" and is also differentiable which guarantees that the surface is differentiable. Thus the surface could be given in the mathematical form:

$$(x, y, f(x, y)),$$

$$z = f(x, y). \tag{3.1}$$

The second expression describes a surface by means of the the coordinate "*z*" is given as a function of the other two and has always a one-to-one orthogonal projection on at least one of the coordinate planes.

3.1 First Fundamental Form of Surfaces

The exact calculation of the curvatures in any direction with any values of the slopes can be derived using the *fundamental forms of surfaces*, which are studied in differential geometry to describe the properties of closed or open surfaces. A closed surface may be for example a surface of a solid body, like a sphere. An open surface may be an optical surface or the wavefront of an optical system. To mathematically describe closed surfaces a vector function $\mathbf{r}(u, v)$ starting at some point and ending at the surface is used. The points over the surface are defined by curvilinear coordinates (u, v) over the surface. For example, a sphere is described by a vector function $\mathbf{r}$ = constant, starting at the center of the sphere [14].

The first fundamental form describes the metric properties of a surface, allowing the calculation of the length of curves, the areas of rectangles, and angles in this surface. It follows from the Pythagorean representation of a differential arc length on a surface. This form is described in terms of the first fundamental form functions, E, F and G, which provide information about the slopes of the surface. If a surface is deformed by bending, but without any stretching, compression, or tearing, the first fundamental form functions remain unchanged.

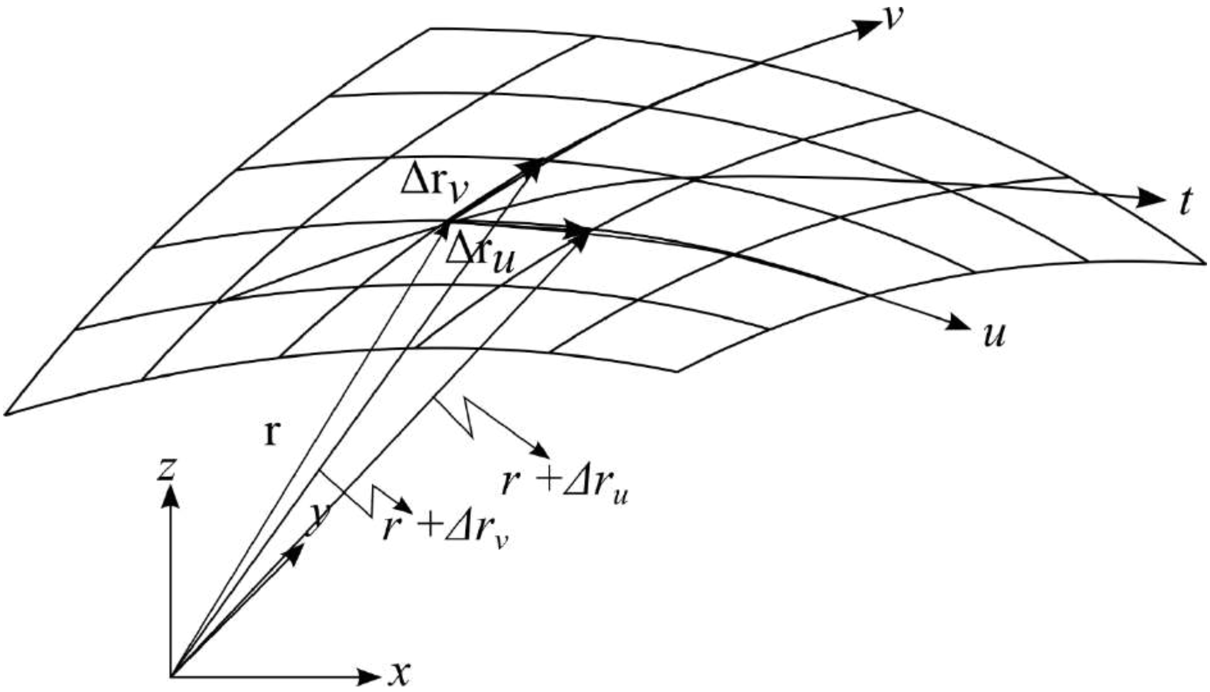

Figure 7. –A surface is defined by the vector function ***r** (u, v)*.

In a general manner, considering open or closed surfaces, we define a surface as in Fig. 7 by locus of points given by a vector function $\mathbf{r}(u, v)$ starting at some common point, which could be at any location, including a point at the infinite. The end of the vector is at a point $\mathbf{P}(u,v)$ on the surface. The origin of the coordinates x, y, z, are at the origin where the vectors $\mathbf{r}$ starts. Thus, the end of these vectors is at the coordinates $x(u, v)$, $y(u, v)$, $z(u,v)$. The coordinates (u, v) could be, but not necessarily, mutually perpendicular at all points on the surface. They are frequently called orthogonal curvilinear coordinates.

As illustrated in Fig 7. Two vectors $\Delta\mathbf{r}_u(u,v)$ and $\Delta\mathbf{r}_v(u,v)$ are added to the vector $\mathbf{r}(u,v)$, over the surface in the directions u and v, obtaining as a first-order approximation with a Taylor series [14]:

Curvature

$$\mathbf{r}(u+du,v+dv)=\mathbf{r}(u,v)+\Delta\mathbf{r}_{\mathrm{u}}(u,v)+\Delta\mathbf{r}_{\mathrm{v}}(u,v)$$
$$=\mathbf{r}(u,v)+\frac{\partial\mathbf{r}(u,v)}{\partial u}\mathrm{d}u+\frac{\partial\mathbf{r}(u,v)}{\partial v}\mathrm{d}v \tag{3.2}$$

with:

$$\Delta\mathbf{r}_u=\frac{\partial\mathbf{r}}{\partial u}\mathrm{d}u \quad \text{and} \quad \Delta\mathbf{r}_v=\frac{\partial\mathbf{r}}{\partial v}\mathrm{d}v \tag{3.3}$$

Thus, $\Delta\mathbf{r}_u$ is in the u direction and $\Delta\mathbf{r}_v$ is in the v direction. They are very small, but their magnitude are different from du and dv.

If a curve is on the surface whose length along the curve in the direction t is s, we can write (t is a parameter, function with coordinates u and v, along the curve):

$$\frac{\partial\mathbf{r}}{\partial t}=\frac{\partial\mathbf{r}}{\partial u}\frac{\partial u}{\partial t}+\frac{\partial\mathbf{r}}{\partial v}\frac{\partial v}{\partial t} \tag{3.4}$$

This is a vector whose square of its modulus is given by the scalar multiplication:

$$\left|\frac{\partial\mathbf{r}}{\partial t}\right|^2=\left(\frac{\partial\mathbf{r}}{\partial u}\frac{\partial u}{\partial t}+\frac{\partial\mathbf{r}}{\partial v}\frac{\partial v}{\partial t}\right)\cdot\left(\frac{\partial\mathbf{r}}{\partial u}\frac{\partial u}{\partial t}+\frac{\partial\mathbf{r}}{\partial v}\frac{\partial v}{\partial t}\right)$$
$$=\frac{\partial\mathbf{r}}{\partial u}\frac{\partial\mathbf{r}}{\partial u}\left(\frac{\partial u}{\partial t}\right)^2+2\frac{\partial\mathbf{r}}{\partial u}\frac{\partial\mathbf{r}}{\partial v}\left(\frac{\partial u}{\partial t}\frac{\partial v}{\partial t}\right)+\frac{\partial\mathbf{r}}{\partial v}\frac{\partial\mathbf{r}}{\partial v}\left(\frac{\partial v}{\partial t}\right)^2 \tag{3.5}$$

Thus, multiplying by dt^2:

$$(\mathrm{d}s)^2=|\,\mathrm{d}\mathbf{r}\,|^2=\left(\frac{\partial\mathbf{r}}{\partial u}\frac{\partial\mathbf{r}}{\partial u}\right)\mathrm{d}u^2+2\left(\frac{\partial\mathbf{r}}{\partial u}\frac{\partial\mathbf{r}}{\partial v}\right)\mathrm{d}u\,\mathrm{d}v+\left(\frac{\partial\mathbf{r}}{\partial v}\frac{\partial\mathbf{r}}{\partial v}\right)\mathrm{d}v^2 \tag{3.6}$$

where ds is a small distance along the curve and the first fundamental functions are:

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$$E = \left(\frac{\partial \mathbf{r}}{\partial u} \frac{\partial \mathbf{r}}{\partial u} \right) = \left| \frac{\partial \mathbf{r}}{\partial u} \right|^2 ; \quad F = \left(\frac{\partial \mathbf{r}}{\partial u} \frac{\partial \mathbf{r}}{\partial v} \right) \cos \theta_{uv} ; G = \left(\frac{\partial \mathbf{r}}{\partial v} \frac{\partial \mathbf{r}}{\partial v} \right) = \left| \frac{\partial \mathbf{r}}{\partial v} \right|^2 \quad (3.7)$$

where θ_{uv} is the angle between the direction of the curvilinear coordinates u and v. Hence, the first fundamental form of the surface can be written:

$$\mathrm{d}s^2 = E\,\mathrm{d}u^2 + 2F\,\mathrm{d}u\,\mathrm{d}v + G\,\mathrm{d}v^2 \quad (3.8)$$

Also:

$$\Delta\mathbf{r}_u \Delta\mathbf{r}_v = |\Delta\mathbf{r}_u| \, |\Delta\mathbf{r}_v| \cos \theta_{uv} \quad (3.9)$$

Thus, if the curvilinear coordinates (u, v) are orthogonal, the function F becomes zero.

The vector product of these two vectors, represented by $\Delta\mathbf{r}_u(u, v) \mathrm{x} \Delta\mathbf{r}_v(u, v)$ is equal to a new vector that is perpendicular to both vectors $\Delta\mathbf{r}_u(u, v)$ and $\Delta\mathbf{r}_v(u, v)$ and hence perpendicular to the surface. The magnitude of this vector is equal to the area of the parallelogram formed by these two vectors $\Delta\mathbf{r}_u(u, v)$ and $\Delta\mathbf{r}_v(u, v)$, as follows:

$$|\Delta\mathbf{r}_u \times \Delta\mathbf{r}_v| = |\Delta\mathbf{r}_u| |\Delta\mathbf{r}_v| \sin \theta_{uv} \quad (3.10)$$

Using Eqs. 3.9 and 3.10, we get the Lagrange identity:

Curvature

$$\left|\Delta\mathbf{r}_u \times \Delta\mathbf{r}_v\right| = \left[\left|\Delta\mathbf{r}_u\right|^2 \left|\Delta\mathbf{r}_v\right|^2 - \left(\Delta\mathbf{r}_u \Delta\mathbf{r}_v\right)\right]^{1/2} \tag{3.11}$$

A vector normal to the surface, which is the *unit normal* vector, is then be given by:

$$\mathbf{N} = \frac{\Delta\mathbf{r}_u \times \Delta\mathbf{r}_v}{\left|\Delta\mathbf{r}_u \times \Delta\mathbf{r}_v\right|} \tag{3.12}$$

By using the Lagrange identity and the definitions of the parameters *E*, *G* and *F*, we obtain:

$$|\Delta r_u \times \Delta r_v| = \left[|\Delta r_u|^2 |\Delta r_v|^2 - \left(\Delta r_u \bullet \Delta r_v\right)^2\right]^{1/2} = \left[EG - F^2\right]^{1/2} \tag{3.13}$$

Hence, the unit normal vector is given by:

$$\mathbf{N} = \frac{\Delta\mathbf{r}_u \times \Delta\mathbf{r}_v}{\left[EG - F^2\right]^{1/2}} \tag{3.14}$$

Therefore, differential element of area is given by:

$$\mathrm{d}A = \left|\Delta\mathbf{r}_u \times \Delta\mathbf{r}_v\right| \mathrm{d}u\,\mathrm{d}v = \left[EG - F^2\right]^{1/2} \mathrm{d}u\,\mathrm{d}v \tag{3.15}$$

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3.2 Second Fundamental Form of Surfaces

Let us assume that we have a surface as illustrated in Fig. 8 with a plane tangent to this surface at the point **P**(u,v). The tangent plane is mathematically defined by the unit normal vector **N**(u,v) as defined above, normal to the surface at the point **P**(u,v). The line **AB** with length $z(u$+du, v+d$v)$ is perpendicular to the tangent plane and the point **B**(u+du, v+dv) is in the vicinity of the point of tangency of the plane. The length of $z(u$+du, v+d$v)$ can be found if we project the segment **PB** over the unit normal vector **N**(u,v), as follows:

$$z(u+\mathrm{d}u, v+\mathrm{d}v) = [\mathbf{r}(u+\mathrm{d}u, v+\mathrm{d}v) - \mathbf{r}(u,v)] \bullet \mathbf{N}(u,v) \tag{3.16}$$

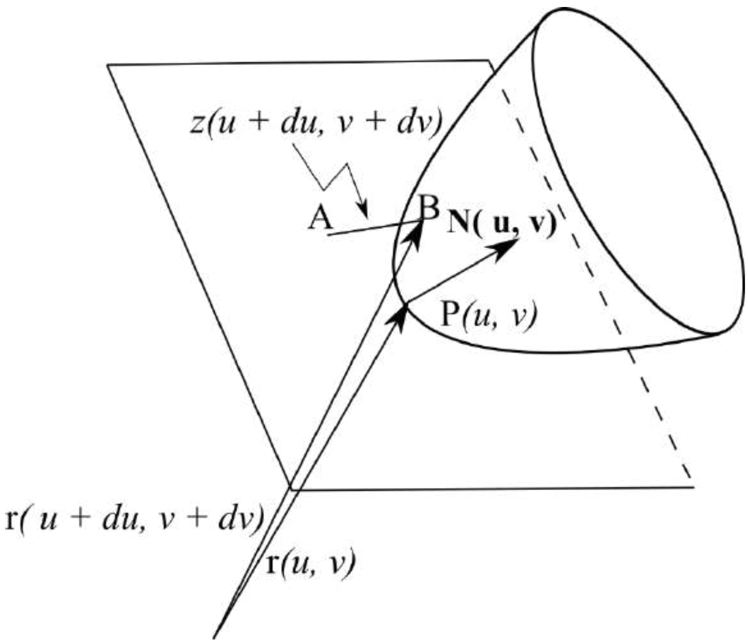

Figure 8. –A plane tangent to the surface and the point **P**.

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Assuming that the surface has continuous derivatives, the vector $\mathbf{r}(u+\mathrm{d}u,v+\mathrm{d}v)$ can be developed about the point **P** in a Taylor series, as in Eqs. 3.2; so that, we can get by Eq 3.16 follows:

$$z(u+\mathrm{d}u,v+\mathrm{d}v)=\frac{1}{2}\left[\frac{\partial^2\mathbf{r}(u,v)}{\partial u^2}\mathbf{N}(u,v)\mathrm{d}u^2+2\frac{\partial^2\mathbf{r}(u,v)}{\partial u\partial v}\mathbf{N}(u,v)\mathrm{d}u\,\mathrm{d}v+\frac{\partial^2\mathbf{r}(u,v)}{\partial v^2}\mathbf{N}(u,v)\mathrm{d}v^2\right]+\ldots \tag{3.17}$$

This expression describes the surface slope's local variation; in other words, it describes how the unit normal vector varies as one moves in a certain direction along the surface. From this information we can obtain knowledge about the second derivatives, in terms of the second fundamental form functions L, M, and N, which are also functions of the first fundamental form functions E, F and G, in the following expression called the second fundamental form:

$$2z(u+\mathrm{d}u,v+\mathrm{d}v)=L\,\mathrm{d}u^2+2M\,\mathrm{d}u\,\mathrm{d}v+N\,\mathrm{d}v^2 \tag{3.18}$$

Hence, we can write:

$$L=\frac{\partial^2\mathbf{r}}{\partial u^2}\mathbf{N};\quad M=\frac{\partial^2\mathbf{r}}{\partial u\partial v}\mathbf{N};\quad N=\frac{\partial^2\mathbf{r}}{\partial v^2}\mathbf{N} \tag{3.19}$$

Or using Eq. 3.14:

$$L=\frac{\partial^2\mathbf{r}}{\partial u^2}\frac{\left(\Delta\mathbf{r}_u\times\Delta\mathbf{r}_v\right)}{\left[EG-F^2\right]^{1/2}};M=\frac{\partial^2\mathbf{r}}{\partial u\partial v}\frac{\left(\Delta\mathbf{r}_u\times\Delta\mathbf{r}_v\right)}{\left[EG-F^2\right]^{1/2}};N=\frac{\partial^2\mathbf{r}}{\partial v^2}\frac{\left(\Delta\mathbf{r}_u\times\Delta\mathbf{r}_v\right)}{\left[EG-F^2\right]^{1/2}}$$

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3.3 Curvatures at a Point in a Surface

Together, the first fundamental form coefficients and the second fundamental form coefficients provide information about the local values of the curvatures. In order to calculate the curvatures along a curve at a position $\mathbf{r}(u, v)$ over a surface, we need to obtain the second derivative of the vector $\mathbf{r}(u, v)$ with respect to s, where the values of the curvilinear coordinates $u(s)$ and $v(s)$, are functions of the position s along the curve. Recalling that the first derivative of the vector $\mathbf{r}(s)$; with respect to s(u, v) is the *tangent vector* to the curve, which can be represented by $\mathbf{t}$, as follows:

$$\mathbf{t} = \frac{\partial \mathbf{r}}{\partial s} = \frac{\partial \mathbf{r}}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial \mathbf{r}}{\partial v}\frac{\partial v}{\partial s} \tag{3.20}$$

and the second derivative of $\mathbf{r}$ with respect to s is:

$$\frac{\partial^2 \mathbf{r}}{\partial s^2} = \frac{\partial \mathbf{r}}{\partial u}\frac{\partial^2 u}{\partial s^2} + \frac{\partial \mathbf{r}}{\partial v}\frac{\partial^2 v}{\partial s^2} + \left(\frac{\partial^2 \mathbf{r}}{\partial u^2}\left(\frac{\partial u}{\partial s}\right)^2 + 2\frac{\partial^2 \mathbf{r}}{\partial u \partial v}\frac{\partial u}{\partial s}\frac{\partial v}{\partial s} + \frac{\partial^2 \mathbf{r}}{\partial v^2}\left(\frac{\partial v}{\partial s}\right)^2 \right) \tag{3.21}$$

The curvature at a point in a surface can be measured along any curve over the surface, but this curve over the surface is not necessarily contained in a plane. However, if a very small section of the curve is taken in the vicinity of the point where the curvature is measured, a tangent plane can be taken to contain this small section of the curve and them measured along this curve in the plane. The magnitude of the change **d*t*** in the tangent vector along the arc distance ds is the curvature c, which can be defined as:

$$c = \left| \frac{\partial^2 \mathbf{r}}{\partial s^2} \right| \tag{3.22}$$

As illustrated in Fig. 8, the plane does not necessarily contain the normal **N** to the surface. In other words, the plane may form an angle with the normal plane. The important curvature should be measured at the same point over the surface, along the curve defined by the intersection of the surface with a normal plane and it is called the *normal curvature*. This curvature is obtained if we take the scalar product of the second derivative of **r**(s) with respect to s, which is a vector contained in the plane, by the unit normal vector **N**. Then, we obtain the projection of the curvature vector over the normal to the surface, obtaining:

$$c = \frac{\partial^2 \mathbf{r}}{\partial s^2}\mathbf{N} = \\ \frac{\partial \mathbf{r}}{\partial u}\mathbf{N}\frac{\partial^2 u}{\partial s^2} + \frac{\partial \mathbf{r}}{\partial v}\mathbf{N}\frac{\partial^2 v}{\partial s^2} + \left(\frac{\partial^2 \mathbf{r}}{\partial u^2}\mathbf{N}\left(\frac{\partial u}{\partial s}\right)^2 + 2\frac{\partial^2 \mathbf{r}}{\partial u \partial v}\mathbf{N}\frac{\partial u}{\partial s}\frac{\partial v}{\partial s} + \frac{\partial^2 \mathbf{r}}{\partial v^2}\mathbf{N}\left(\frac{\partial v}{\partial s}\right)^2 \right) \tag{3.23}$$

The first derivatives of **r** with respect to u and v are orthogonal to the unit normal vector N, so that their scalar product is zero, remaining only the last three terms:

$$c = \frac{\partial^2 \mathbf{r}}{\partial s^2}\mathbf{N} = \frac{\partial^2 \mathbf{r}}{\partial u^2}\mathbf{N}\left(\frac{\partial u}{\partial s}\right)^2 + 2\frac{\partial^2 \mathbf{r}}{\partial u \partial v}\mathbf{N}\frac{\partial u}{\partial s}\frac{\partial v}{\partial s} + \frac{\partial^2 \mathbf{r}}{\partial v^2}\mathbf{N}\left(\frac{\partial v}{\partial s}\right)^2 \tag{3.24}$$

On the other hand, writing the second derivative of **r**(s) in terms of the second fundamental form coefficients by using Eq. (3.19):

$$c = \frac{\partial^2 \mathbf{r}}{\partial s^2}\mathbf{N} = L\left(\frac{\partial u}{\partial s}\right)^2 + 2M\frac{\partial u}{\partial s}\frac{\partial v}{\partial s} + N\left(\frac{\partial v}{\partial s}\right)^2 \tag{3.25}$$

Now, we finally can write this expression as:

$$c = \left|\frac{\partial^2 \mathbf{r}}{\partial s^2}\right| \cos\alpha = \frac{\left[L\,\mathrm{d}u^2 + 2M\,\mathrm{d}u\,\mathrm{d}v + N\,\mathrm{d}v^2\right]}{\mathrm{d}s^2} \tag{3.26}$$

In this expression cos α is the cosine of the angle between the normal plane and the plane, that is, between the vectors **N** and $\mathbf{r}_M$ as described in page 22.

Now, since the arc length *ds*, from the first fundamental form of a surface, is given by Eq. 3.24, we obtain the normal curvature c_θ as:

$$c_\theta = \frac{\left(L\,\mathrm{d}u^2 + 2M\,\mathrm{d}u\,\mathrm{d}v + N\,\mathrm{d}v^2\right)}{\left(E\,\mathrm{d}u^2 + 2F\,\mathrm{d}u\,\mathrm{d}v + G\,\mathrm{d}v^2\right)} \tag{3.27}$$

If we divide all terms in this expression by dv, we see that the ratio dv/du, is the tangent of the angle θ as follows:

$$\tan\theta = \frac{\mathrm{d}v}{\mathrm{d}u} \tag{3.28}$$

thus, obtaining:

$$c_\theta = \frac{\left(L\cos^2\theta + 2M\sin\theta\cos\theta + N\sin^2\theta\right)}{\left(E\cos^2\theta + 2F\sin\theta\cos\theta + G\sin^2\theta\right)} \tag{3.29}$$

3.4 Principal Curvatures

The extreme (maximum and minimum) curvatures values for different directions are called principal curvatures. If the slopes along with the x and y coordinates, at the point where the curvatures are evaluated are equal to zero, in other words, if the tangent plane at that point is parallel to the x-y plane, then equation (3.29) becomes as follows.

The numerator in the expression for the curvature c_θ in Eq. 3.27, is:

$$\frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}\right)+\frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}-\frac{\partial^2 f}{\partial y^2}\right)\cos 2\theta+\frac{\partial^2 f}{\partial x \partial y}\sin 2\theta \tag{3.30}$$

The principal directions are obtained with the condition:

$$\frac{\partial c_\theta}{\partial \theta}=0 \tag{3.31}$$

obtaining:

$$-\left(\frac{\partial^2 f}{\partial x^2}-\frac{\partial^2 f}{\partial y^2}\right)\sin 2\theta+2\frac{\partial^2 f}{\partial x \partial y}\cos 2\theta=0 \tag{3.32}$$

or:

$$\tan 2\theta=\frac{2\dfrac{\partial^2 f}{\partial x \partial y}}{\left(\dfrac{\partial^2 f}{\partial x^2}-\dfrac{\partial^2 f}{\partial y^2}\right)} \tag{3.33}$$

Hence, the principal curvatures κ_1 and κ_2, will be obtained by evaluating them using Eq. 3.30 with these principal directions.

Using the first and second fundamental form of the surfaces, the principal curvatures κ_1 and κ_2, can also be found in Stoker as it will be described here [9, p. 91]. Since the curvature is a function of the angle about the point **P** where the curvature is measured, if we go around this point excluding the center at du = dv = 0 we can calculate the maximum and the minimum values of this curvatures. Thus, these extreme values of κ can be found by setting:

$$\frac{\partial \kappa}{\partial (\mathrm{d}u)} = 0; \qquad \frac{\partial \kappa}{\partial (\mathrm{d}v)} = 0 \tag{3.34}$$

obtaining:

$$\left(L - \kappa E\right)\mathrm{d}u + \left(M - \kappa F\right)\mathrm{d}v = 0 \tag{3.35}$$

and:

$$\left(M - \kappa F\right)\mathrm{d}u + \left(N - \kappa G\right)\mathrm{d}v = 0 \tag{3.36}$$

Now, these two equations can be expressed as:

$$\begin{vmatrix} \left(L - \kappa E\right) & \left(M - \kappa F\right) \\ \left(M - \kappa F\right) & \left(N - \kappa G\right) \end{vmatrix} = 0 \tag{3.37}$$

From this determinant we then can write the following second order equation:

$$\left(EG - F^2\right)\kappa^2 + \left(EN + GL - 2FM\right)\kappa + \left(LN - M^2\right) = 0 \tag{3.38}$$

This expression can be written as [10]:

$$\kappa^2 - 2H\kappa + K = 0 \tag{3.39}$$

where H and K are the second fundamental coefficients, representing the main or average curvature and the Gaussian curvature, respectively, that is $H = k_1 k_2$ and $K = k_1 + k_2$

The principal curvatures are the roots of this quadratic equation. Substituting the expressions for H and K we obtain:

$$\kappa^2 - \frac{(EN + GL - 2FM)}{(EG - F^2)}\kappa + \frac{(LN - M^2)}{(EG - F^2)} = 0 \tag{3.40}$$

Hence, the two principal curvatures are:

$$\kappa = H \pm \sqrt{H^2 - K^2} \tag{3.41}$$

or:

$$\kappa = \frac{(EN + GL - 2FM) \pm \sqrt{(EN + GL - 2FM)^2 - 4(LN - M^2)^2}}{2(EG - F^2)} \tag{3.42}$$

To obtain the principal directions, where the principal values are, we can write from Eqs. 3.38 and 3.39:

$$(L\,\mathrm{d}u + M\,\mathrm{d}v) - \kappa(E\,\mathrm{d}u + F\,\mathrm{d}v) = 0 \tag{3.43}$$

and:

$$\left(M\,\mathrm{d}u + N\,\mathrm{d}v\right) - \kappa\left(F\,\mathrm{d}u + G\,\mathrm{d}v\right) = 0 \tag{3.44}$$

This system can have a non-trivial solution if and only if the following determinant is equal to zero:

$$\begin{vmatrix} L\,\mathrm{d}u + M\,\mathrm{d}v & E\,\mathrm{d}u + F\,\mathrm{d}v \\ M\,\mathrm{d}u + N\,\mathrm{d}v & F\,\mathrm{d}u + G\,\mathrm{d}v \end{vmatrix} = 0 \tag{3.45}$$

Then, by developing the determinant, the orientation of the principal planes in the quadratic equation becomes:

$$\left(FN - MG\right)\left(\frac{\mathrm{d}v}{\mathrm{d}u}\right)^2 + \left(EN - LG\right)\left(\frac{\mathrm{d}v}{\mathrm{d}u}\right) + \left(EM - LF\right) = 0 \tag{3.46}$$

Thus, the principal directions are:

$$\tan\theta = \left(\frac{\mathrm{d}v}{\mathrm{d}u}\right) = \frac{-(EN-LG)\pm\sqrt{(EN-LG)^2-4(FN-LG)(EN-LF)}}{2(FN-MG)} \tag{3.47}$$

The principal curvatures in Cartesian coordinates are the two solutions to the following second order equation:

$$\kappa^2 - \frac{\left[1+\left(\frac{\partial f}{\partial x}\right)^2\right]\left(\frac{\partial^2 f}{\partial y^2}\right)+\left[1+\left(\frac{\partial f}{\partial y}\right)^2\right]\left(\frac{\partial^2 f}{\partial x^2}\right)-2\left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial f}{\partial y}\right)\left(\frac{\partial^2 f}{\partial x \partial y}\right)}{\left(1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2\right)^{3/2}}\kappa + \frac{\left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right)-\left(\frac{\partial^2 f}{\partial xy}\right)^2}{\left(1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2\right)^2}=0 \tag{3.48}$$

and the principal directions are the two solutions to the following second order equation:

$$\begin{aligned}&\left[\left(\frac{\partial^2 f}{\partial y^2}\right)\left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial f}{\partial y}\right)-\left(\frac{\partial^2 f}{\partial xy}\right)\left(1+\left(\frac{\partial f}{\partial y}\right)^2\right)\right]\tan^2\phi\\&+\left[\left(\frac{\partial^2 f}{\partial y^2}\right)\left(1+\left(\frac{\partial f}{\partial x}\right)^2\right)-\left(\frac{\partial^2 f}{\partial x^2}\right)\left(1+\left(\frac{\partial f}{\partial y}\right)^2\right)\right]\tan\phi\\&+\left[\left(\frac{\partial^2 f}{\partial xy}\right)\left(1+\left(\frac{\partial f}{\partial y}\right)^2\right)-\left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial f}{\partial y}\right)\right]=0\end{aligned} \tag{3.49}$$

Solving these two second order equations the principal curvatures, as well as the principal directions, can be found.

Besides the previously described curvatures, in differential geometry, the *mean* curvature c_{av} and the *Gaussian* curvature c_g have been defined as the arithmetic average and the product, respectively, of the two principal curvatures, as follows:

$$c_{av}=\frac{\kappa_1+\kappa_2}{2} \qquad \text{and} \qquad c_g=\kappa_1\kappa_2 \tag{3.50}$$

3.5 First and Second Fundamental Form Coefficients in Cartesian Coordinates

The first and second fundamental forms apply to open as well as to closed surfaces, for example, the surface of a volume. Let us now restrict ourselves to open surfaces, so that given any point with coordinates (*x, y*) there is only one possible value of the function *f*(*x, y*) describing the surface. Then, we can apply the first and second fundamental forms to these surfaces by describing the surface in Cartesian coordinates. We use the Monge parametrization, frequently used when the surface deviates only weakly from a plane, although we can use it for any open surface. The height of the surface at the point (*x*, *y*) in the close plane is given by *f*(*x*, *y*). Then, in this case the position vector coordinates for the vector **r** are (*x*, *y*, *f*(*x*,*y*)) [9] . The derivatives with respect to the coordinates *u* and *v* are replaced by the derivatives with respect to *x* and *y*. With this representation, the fundamental first form coefficients in Cartesian coordinates are given by:

$$
\begin{aligned}
E &= \left(\frac{\partial \mathbf{r}}{\partial x} \bullet \frac{\partial \mathbf{r}}{\partial x}\right) = \left(\left(1,\ 0,\ \frac{\partial f}{\partial x}\right) \bullet \left(1,\ 0,\ \frac{\partial f}{\partial x}\right)\right) \\
F &= \left(\frac{\partial \mathbf{r}}{\partial x} \bullet \frac{\partial \mathbf{r}}{\partial y}\right) = \left(\left(1,\ 0,\ \frac{\partial f}{\partial x}\right) \bullet \left(0,\ 1,\ \frac{\partial f}{\partial y}\right)\right) \\
G &= \left(\frac{\partial \mathbf{r}}{\partial y} \bullet \frac{\partial \mathbf{r}}{\partial y}\right) = \left(\left(0,\ 1,\ \frac{\partial f}{\partial y}\right) \bullet \left(0,\ 1,\ \frac{\partial f}{\partial y}\right)\right)
\end{aligned} \tag{3.53}
$$

Thus, obtaining in Cartesian coordinates:

$$
\begin{aligned}
E &= 1 + \left(\frac{\partial f}{\partial x}\right)^2 \\
F &= \left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial f}{\partial y}\right) \\
G &= 1 + \left(\frac{\partial f}{\partial y}\right)^2
\end{aligned} \tag{3.54}
$$

Similarly, the fundamental second form coefficients in Cartesian coordinates are given by:

$$
\begin{aligned}
L &= \frac{\dfrac{\partial^2 \mathbf{r}}{\partial x^2} \bullet \left(\dfrac{\partial \mathbf{r}}{\partial x} \times \dfrac{\partial \mathbf{r}}{\partial y} \right)}{\left(EG - F^2 \right)^{1/2}} = \frac{\left(0,\, 0,\, \dfrac{\partial^2 f}{\partial x^2} \right) \bullet \left(-\dfrac{\partial f}{\partial x}, -\dfrac{\partial f}{\partial y}, 1 \right)}{\left(EG - F^2 \right)^{1/2}} \\
M &= \frac{\dfrac{\partial^2 \mathbf{r}}{\partial x \partial y} \bullet \left(\dfrac{\partial \mathbf{r}}{\partial x} \times \dfrac{\partial \mathbf{r}}{\partial y} \right)}{\left(EG - F^2 \right)^{1/2}} = \frac{\left(0,\, 0,\, \dfrac{\partial^2 f}{\partial x \partial y} \right) \bullet \left(-\dfrac{\partial f}{\partial x}, -\dfrac{\partial f}{\partial y}, 1 \right)}{\left(EG - F^2 \right)^{1/2}} \\
N &= \frac{\dfrac{\partial^2 \mathbf{r}}{\partial y^2} \bullet \left(\dfrac{\partial \mathbf{r}}{\partial x} \times \dfrac{\partial \mathbf{r}}{\partial y} \right)}{\left(EG - F^2 \right)^{1/2}} = \frac{\left(0,\, 0,\, \dfrac{\partial^2 f}{\partial y^2} \right) \bullet \left(-\dfrac{\partial f}{\partial x}, -\dfrac{\partial f}{\partial y}, 1 \right)}{\left(EG - F^2 \right)^{1/2}}
\end{aligned}
\tag{3.55}
$$

Thus, obtaining the second fundamental form coefficients in Cartesian coordinates as:

$$
\begin{aligned}
L &= \frac{\dfrac{\partial^2 f}{\partial x^2}}{\left(EG - F^2 \right)^{1/2}} \\
M &= \frac{\dfrac{\partial^2 f}{\partial x \partial y}}{\left(EG - F^2 \right)^{1/2}} \\
N &= \frac{\dfrac{\partial^2 f}{\partial y^2}}{\left(EG - F^2 \right)^{1/2}}
\end{aligned}
\tag{3.56}
$$

The square of the differential arc length in terms of the functions E, F and G can be written as:

$$
\mathrm{d}s^2 = \left(1 + \left(\frac{\partial f}{\partial x} \right)^2 \right) \mathrm{d}x^2 + 2 \frac{\partial f}{\partial x} \frac{\partial f}{\partial y} \mathrm{d}x\, \mathrm{d}y + \left(1 + \left(\frac{\partial f}{\partial y} \right)^2 \right) \mathrm{d}y^2
\tag{3.57}
$$

Curvatures

and the discriminant ($EG - F^2$):

$$\left(EG - F^2\right) = \left(1 + \left(\frac{\partial f}{\partial x}\right)^2\right)\left(1 + \left(\frac{\partial f}{\partial y}\right)^2\right) - \left(\frac{\partial f}{\partial x}\right)^2\left(\frac{\partial f}{\partial y}\right)^2$$
$$= 1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 \tag{3.58}$$

which is equal to one if the slopes at the origin are close to zero.

This discriminant can be used to describe the mean and Gaussian curvatures. On a surface, different values of the Gaussian curvature characterize points with different names, as in Table 3.1.

Table 3.1.- Name of points in surface, with different values of the Gaussian curvature or the sign of the discriminant $LN - M^2$. The denominator $EG - F^2$ is always positive. As pointed out before, F is zero if and only if the curvilinear coordinates are orthogonal.

Gaussian curvature value	*Discriminant* $LN - M^2$	*Name of point in the surface*
Negative	< 0	Hyperbolic (Saddle)
Zero	$= 0$	Parabolic or planar
Positive	> 0	Elliptic

A differential element of area is given by:

$$\begin{aligned} \mathrm{d}A &= \left[EG - F^2\right]^{1/2} \mathrm{d}x\,\mathrm{d}y \\ &= \left[1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right]^{1/2} \mathrm{d}x\,\mathrm{d}y \end{aligned} \tag{3.59}$$

The expression for the normal curvature in the θ direction, from Eq. (3.28) is:

$$c_\theta = \frac{\dfrac{\partial^2 f}{\partial x^2}\cos^2\theta + \dfrac{\partial^2 f}{\partial y^2}\sin^2\theta + 2\dfrac{\partial^2 f}{\partial x \partial y}\sin\theta\cos\theta}{\left(1 + \left(\dfrac{\partial f}{\partial x}\cos\theta + \dfrac{\partial f}{\partial y}\sin\theta\right)^2\right)\left(1 + \left(\dfrac{\partial f}{\partial x}\right)^2 + \left(\dfrac{\partial f}{\partial y}\right)^2\right)^{1/2}} \tag{3.60}$$

To fully describe the concept of local curvature in a surface, let us first begin by considering a one-dimensional function $f(x)$ represented in a plane, as in Fig. 9, where three circles are drawn close to the curve.

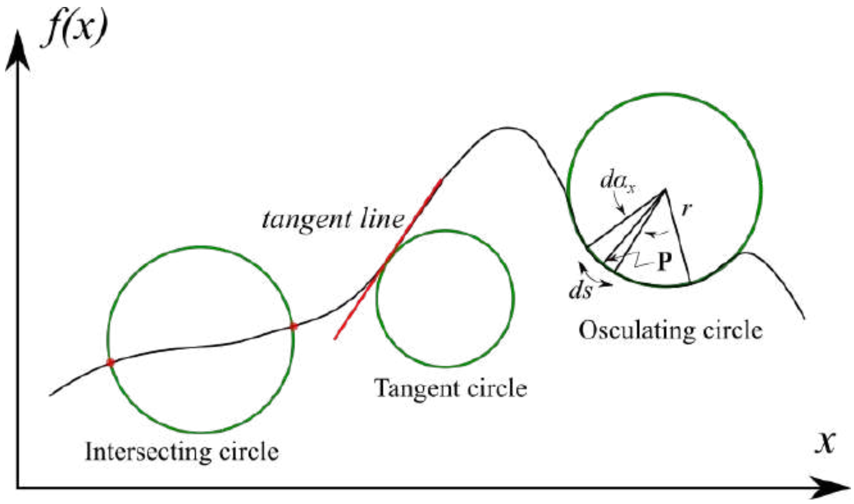


Figure 9. – Three circles near a curve. From left to the right, intersected, tangent and osculant.

The first circle at the left, in Fig. 9 is intersected by the curve at two points. The second circle touches the curve at only one point, where the first derivative (slope) at this point is the same both at the curve and at the circle. At a given point in the curve, we can trace an infinite number of tangent circles with different sizes, on any side of the curve. The third circle touches the curve in a small region, where both the first and the second derivatives are the same at the center of the region where they touch each other. At a given point in the curve, we can trace only one of these circles. Then, we say that the circle and the curve are osculating (from Latin: *osculum* = kiss). The curve at this small region has the same radius of curvature as the *osculating circle* and for this one-dimensional case, x-$f(x)$ is also the *osculating plane*. The unit normal vector is a vector passing through the point in the curve being considered and pointing to the center of curvature of the osculating circle. The curvature is defined by the inverse of the radius of curvature r. More formally, we may say that for a small length along with the curve ds, the curvature is given by how fast the slope is changing at the point **P** in Fig 9, then, as defined probably for the first time by Kastner in (1759), the curvature along with the *x-direction* c_x at the point **P** would then be expressed as:

$$c_x = \frac{\mathrm{d}\alpha_x}{\mathrm{d}s} \tag{3.58}$$

where ds is the small length traveled along the curve and dα is the change in the slope of the curve along the *x-axis*. On the other hand, the slope at the same point **P** is given by the derivative of $f(x)$ with respect to x:

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = \tan\alpha_x \tag{3.59}$$

The second derivative of the function $f(x)$ with respect to x is equal to the curvature only if the slope at that point is zero. Otherwise, the curvature can be found by writing the second derivative at any point where the first derivative (slope angle equal to α_x) is not zero, as follows:

$$\frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2} = \frac{\mathrm{d}\tan\alpha_x}{\mathrm{d}x} = \sec^2\alpha_x \frac{\mathrm{d}\alpha_x}{\mathrm{d}x} = \frac{1}{\cos^2\alpha_x}\frac{\mathrm{d}\alpha_x}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}x} = \frac{1}{\cos^3\alpha_x} c_x \tag{3.60}$$

Hence, the curvature, measured along a curve on a flat surface, is given by:

$$c_x = \frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2}\cos^3\alpha_x \tag{3.61}$$

which, by writing cos α_x in terms of the tan α_x, which is the first derivative of f with respect to x can be written as:

$$c_x = \frac{\dfrac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2}}{\left(1+\left(\dfrac{\mathrm{d}f(x)}{\mathrm{d}x}\right)^2\right)^{3/2}} \tag{3.62}$$

This result was found with a different notation by Bernoulli (1691) and more formally some years later by Newton (1736). This curvature is a function of both the first and the second derivative. If the first derivative is zero, the curvature in the *x-direction* is just the second derivative of $f(x)$. Thus, we have found expressions to obtain the curvature at a point along a curve $f(x)$ contained in a plane, and the normal to this curve is also contained in the same plane [18].

This result can also be applied to find the exact value of the local curvature over a curve on a surface $f(x, y)$ in the direction of maximum slope (gradient), where the slope in the perpendicular direction is zero. In the next section, we will describe how to find the local curvature along a curve on the surface $f(x, y)$ in the direction of no slope, perpendicularly to the direction of maximum slope.

4 Local Curvatures

4.1 Local curvatures in a Perpendicular direction to its maximum Slope direction

The concept of curvature was extended by Euler (1767), Bernoulli's doctoral student, to three dimensions. In a more general case we have a surface $f(x, y)$, and a curve on this surface, this expression is strictly valid if at the points $\mathbf{P}(x, y)$ where the curvature is to be evaluated has a slope along some direction of the curve but not in the perpendicular direction. We will study the local curvatures over an open surface $f(x, y)$, so that given any point $\mathbf{P}$ with coordinates (x, y) there is only one possible value of the function $f(x, y)$ describing the surface. Let us start by defining some important concepts. A *normal plane* at a point in a surface is any plane containing the normal to the surface at that point. The intersection of a normal plane with the surface is a curve called a *normal section* and the curvature of this curve at that point is the *normal curvature*.

At any point on the surface, we can place a plane tangent at that point. This plane is not necessarily horizontal, which in the general case has a tilt. At the point of tangency on this surface, there is a different slope for different directions, which is equal to the first derivative in the given direction. Thus, at this point, there are two mutually perpendicular directions, one with a zero slope and one with a maximum slope (gradient). It should be pointed out that these zero and maximum slope directions are not necessarily the same as the maximum and minimum curvatures, which are also called the *principal curvatures*. However, it is interesting to notice two particular cases: a) If the surface has rotational symmetry, at any off-axis point, the principal curvatures are one in the direction of maximum slope (gradient) and one in the direction of zero slopes. In these surfaces, these are the radial and the angular directions. b) A second interesting case is a surface with symmetry about a straight line in the *x-y* plane, like for example, a horizontal cylinder. In this case, the principal curvatures are also one in the direction of maximum slope (gradient) and one in the direction of no slope.

The local curvatures at the point in the surface being considered can be easily calculated in these two special directions, the direction of maximum slope and the direction of zero slopes. For the direction of the maximum slope, there is no slope in the perpendicular direction, and hence the theory in Chapter. 4.1 and the result in Eq. 3.62 can be applied if the *x-direction* is selected in the direction of maximum slope. Next, we will calculate the local curvature in the direction of zero slopes.

Figure 10 illustrates a single-valued function $f(x,y)$, describing a surface on top of a sphere and a cylinder. Let us assume that on this surface we can find two points $\mathbf{P}_1$ and $\mathbf{P}_2$, with the following characteristics. At point $\mathbf{P}_1$ the local curvatures on this point in the surface $f(x, y)$ are the same in all directions and thus, we can place an osculating sphere at this point. At the point $\mathbf{P}_2$ in the surface $f(x, y)$ the curvature along the x coordinate is zero, hence the axis of the cylinder is contained in a plane perpendicular to the y axis. Thus, we can place an osculating cylinder located at this point $\mathbf{P}_2$. The *unit normal vectors* $\mathbf{N}$ are at the points where the local curvatures are to be measured. The unit vectors $\mathbf{r}_M$, also at the points $\mathbf{P}_1$ and $\mathbf{P}_2$, are parallel to the *z-axis*. Thus, the angle between these two vectors $\mathbf{N}$ and $\mathbf{r}_M$ is equal to the angle α_g formed by the maximum slope (gradient) on the surface.

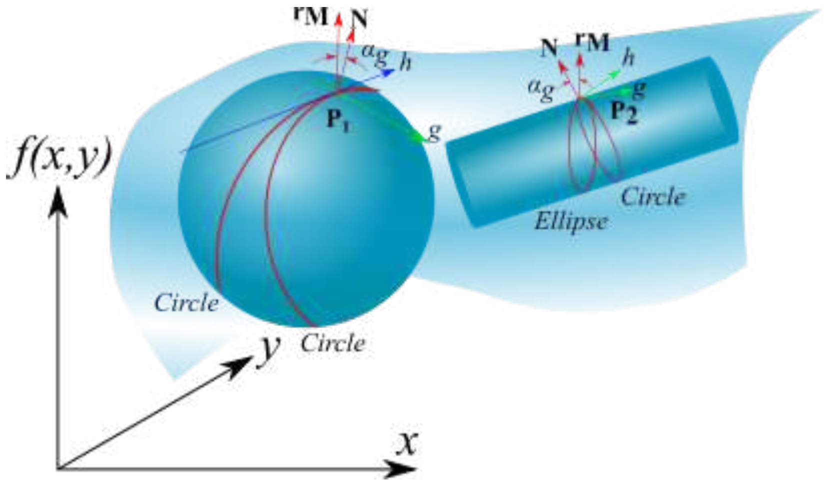


Figure 10. - A surface $f(x,y)$ with an osculating sphere at the point P_1 and an osculating cylinder at the point P_2. The curvature at the two points P_1 and P_2, along the direction of h with no slope is to be measured and the maximum slope is in the direction of g.

With a simple geometrical analysis, it is relatively simple to show that in general, for the osculating sphere as well as for the osculating cylinder, that the curvature in the direction of no slope, which the perpendicular direction to that of maximum slope (called here direction h), at a point located along a curve in the plane containing the normal to the surface would be given by the second derivative in this direction, multiplied by cos α_g. Thus, if the slope along the curve in the direction h is zero, but different from zero in the perpendicular direction g, the curvature c_h is:

$$c_h = \frac{\partial^2 f}{\partial h^2} \cos\alpha_g \tag{4.1}$$

This result is known as the theorem of Meusnier, to honor a mathematical genius that died just before his 39th birthday in Napoleon's army in 1793, after writing his monumental work in differential geometry [19]. Writing cos α_g in terms of the tan α_g, which is the first derivative of f with respect to g:

$$c_h = \frac{\dfrac{\partial^2 f}{\partial h^2}}{\left(1 + \left(\dfrac{\partial f}{\partial g}\right)^2\right)^{1/2}} \tag{4.2}$$

In conclusion, we can find the exact local curvatures values in the directions of maximum slope and no slope, using Eqs, 4.1 and 4.2 respectively.

4.2 Curvatures in Different Directions at a Point on a Surface

We have calculated the local curvature at a point in a surface, in two particular mutually perpendicular directions, the direction of maximum slope and the direction of no slope. In any other direction, the problem is mathematically more complicated, but the problem has been solved by differential geometry methods, by the development of the so called first fundamental and the second fundamental form of surfaces, as pointed out before. However, these two expressions are enough to obtain a highly accurate and intuitive method for calculating and understanding the local curvatures as described below.

Let us consider a point in a surface $f(x, y)$ and an infinite number of possible directions for the curvature, passing through a point, as illustrated in Fig. 11.

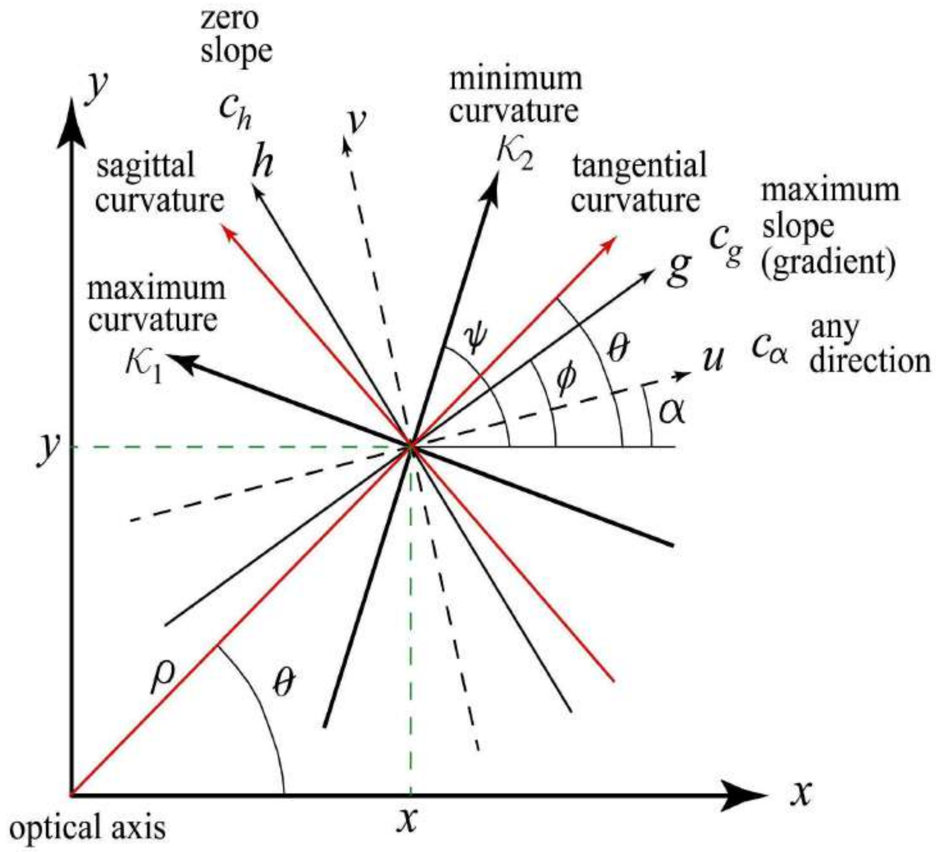


Figure 11. –Different orientations for the zero and maximum slopes and the principal curvatures at a point in a surface.

For this point in this surface, the slopes in the x and y directions are given by:

$$\frac{\partial f(x,y)}{\partial x} \quad \text{and} \quad \frac{\partial f(x,y)}{\partial y} \tag{4.3}$$

The gradient of the function $f(x, y)$ is a vector in the direction of maximum slope, in the direction of g, as illustrated in Fig. 11, whose magnitude is given by:

$$|\nabla f(x,y)| = \frac{\partial f(x,y)}{\partial g} = \left[\left(\frac{\partial f(x,y)}{\partial x}\right)^2 + \left(\frac{\partial f(x,y)}{\partial y}\right)^2\right]^{1/2} \tag{4.4}$$

and the angle ϕ with respect to the x axis for the direction g is given by:

Curvature

$$\tan\phi = \frac{\left(\frac{\partial f(x,y)}{\partial y}\right)}{\left(\frac{\partial f(x,y)}{\partial x}\right)} \tag{4.5}$$

Sometimes, it is necessary to find the values of cos 2ϕ and sin 2ϕ , for the gradient direction, which can be shown to be:

$$\cos 2\phi = \frac{\left(\frac{\partial f}{\partial x}\right)^2 - \left(\frac{\partial f}{\partial y}\right)^2}{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} : \quad \sin 2\phi = 2\frac{\left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial f}{\partial y}\right)}{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} \tag{4.6}$$

The local curvature measured at a point over a surface $f(x, y)$, along the direction of the gradient with the maximum surface slope tan α_g will be represented by c_g and it is given by

$$c_g = \frac{\partial^2 f(x,y)}{\partial g^2}\cos^3\alpha_g \tag{4.7}$$

The first derivative of $f(x, y)$ with respect to g is:

$$\tan\alpha_g = \frac{\partial f(x,y)}{\partial g} \tag{4.8}$$

Then, writing cosα_g in terms of the tan α_g, we obtain:

$$c_g = \frac{\frac{\partial^2 f(x,y)}{\partial g^2}}{\left(1 + \left(\frac{\partial f(x,y)}{\partial g}\right)^2\right)^{3/2}} \tag{4.9}$$

Curvatures

Thus, formally probing the use of Eq. 4.9 to find the local curvature along the gradient, as pointed out before.

4.2 Curvatures in a Surface *f*(*x, y*) in any Direction *α*

Now, to calculate the local curvatures at any point in any desired direction *α* (See Figs. 11 and 12) we need to know the first and second derivatives at the desired location. Let us assume that we need to calculate the first and second derivatives at the point (*x*, *y*) but in the direction *α* of the rotated coordinates *u* and *v*, as illustrated in Fig. 12:

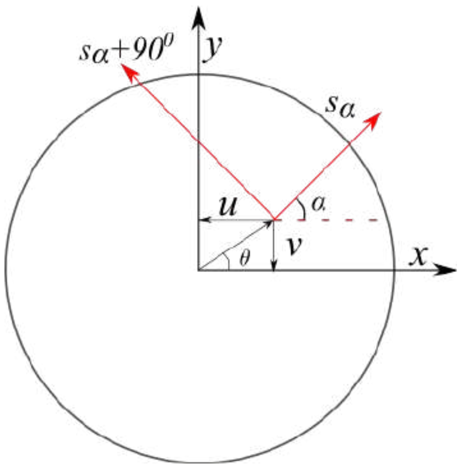


Figure 12. - Translation and rotation of coordinates to evaluate the local curvatures at the point (*x*, *y*) along the rotated axis *u* in the direction *α*.

The function representing the surface is *f*(*x, y*). The first derivative with respect to the coordinate *u*, in the *α* direction is:

$$\frac{\partial f(x,y)}{\partial u} = \frac{\partial f}{\partial x}\cos\alpha + \frac{\partial f}{\partial y}\sin\alpha \tag{4.10}$$

and the first derivative with respect to the coordinate *v* in the $\alpha + 90^0$ direction is:

Curvature

$$\frac{\partial f(x,y)}{\partial v} = -\frac{\partial f}{\partial x}\sin\alpha + \frac{\partial f}{\partial y}\cos\alpha \tag{4.11}$$

The second derivative of $f(x,y)$ with respect to the coordinate u is given by:

$$\begin{aligned}
\frac{\partial^2 f(x,y)}{\partial u^2} &= \frac{\partial}{\partial u}\left(\cos\alpha\frac{\partial f}{\partial x} + \sin\alpha\frac{\partial f}{\partial y}\right) \\
&= \left(\cos\alpha\frac{\partial}{\partial x} + \sin\alpha\frac{\partial}{\partial y}\right)\left(\cos\alpha\frac{\partial f}{\partial x} + \sin\alpha\frac{\partial f}{\partial y}\right) \\
&= \frac{\partial^2 f}{\partial x^2}\cos^2\alpha + \frac{\partial^2 f}{\partial y^2}\sin^2\alpha + 2\frac{\partial^2 f}{\partial x\partial y}\sin\alpha\cos\alpha \\
&= \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) + \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2}\right)\cos 2\alpha + \frac{\partial^2 f}{\partial x\partial y}\sin 2\alpha
\end{aligned} \tag{4.12}$$

This second derivative along the u axis, in α the direction given by this expression is equal to the curvature c_θ only when the slopes at the point in the surface where this curvature is to be evaluated are zero.

Now, let us now assume that this surface normal is not perpendicular to the x-y plane. We assume there is a slope $\tan\alpha_u$ along the curve where the curvature is measured, given by the first derivative with respect to the coordinate u, in the α direction as:

$$\tan\alpha_u = \frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\cos\alpha + \frac{\partial f}{\partial y}\sin\alpha \tag{4.13}$$

If besides this inclination of the surface normal in the measurement direction, there is also a slope of the surface or inclination α_v of the surface normal in the perpendicular direction, we might intuitively try to generalize our two curvature expressions (Eqs, 4.7 and 4.13) by writing:

Curvatures

$$c_\alpha = \frac{\partial^2 f(x,y)}{\partial u^2} \cos^3 \alpha_u \cos \alpha_v \tag{4.14}$$

Then, by writing the cos α_u and cos α_v in terms of the slopes we can find:

$$c_\alpha = \frac{\frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) + \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2}\right)\cos 2\alpha + \frac{\partial^2 f}{\partial x \partial y}\sin 2\alpha}{\left[1 + \left(\frac{\partial f}{\partial x}\cos\alpha + \frac{\partial f}{\partial y}\sin\alpha\right)^2\right]^{3/2}\left[1 + \left(\frac{\partial f}{\partial x}\sin\alpha - \frac{\partial f}{\partial y}\cos\alpha\right)^2\right]^{1/2}} \tag{4.15}$$

This expression was derived here in an intuitive manner and it is exact only in the directions of the gradient $\theta = \phi$ (and $\theta = \phi + 180^0$) and perpendicular to the gradient $\theta = \phi + 90^0$ (and $\theta = \phi + 270^0$) but it is only approximate although, highly accurate in all other directions, unless the slopes are zero.

The exact formula derived rigorously from the fundamental forms of the differential geometry (Stoker 1989) is quite similar, as follows:

$$c_\alpha = \frac{\frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) + \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2}\right)\cos 2\alpha + \frac{\partial^2 f}{\partial x \partial y}\sin 2\alpha}{\left(1 + \left(\frac{\partial f}{\partial x}\cos\alpha + \frac{\partial f}{\partial y}\sin\alpha\right)^2\right)\left(1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right)^{1/2}} \tag{4.16}$$

Its derivation is made with curvilinear coordinates, obtaining the first and second fundamental forms of surfaces in three dimensional space and at the end a conversion to Cartesian coordinates with a Monge parametrization, (Stoker 1969, Chap 4) frequently used when the surface does not deviate much from a plane, is applied. Observing this expression, we may notice that the whole denominator becomes equal to one if the first derivative is extremely small or zero. Then, the curvature is just the numerator, which is equal to the second derivative in the direction of the measured curvature.

If the expression is converted from Cartesian to polar coordinates, the exact expression for the local curvatures at the point (ρ, θ) becomes:

$$c_\alpha = \frac{\frac{1}{2}\left(\frac{\partial^2 f}{\partial \rho^2}+\frac{1}{\rho}\frac{\partial f}{\partial \rho}+\frac{1}{\rho^2}\frac{\partial^2 f}{\partial \theta^2}\right)+\frac{1}{2}\left(\frac{\partial^2 f}{\partial \rho^2}-\frac{1}{\rho}\frac{\partial f}{\partial \rho}-\frac{1}{\rho^2}\frac{\partial^2 f}{\partial \theta^2}\right)\cos 2(\theta-\alpha)-\frac{1}{\rho}\left(\frac{\partial^2 f}{\partial \rho \partial \theta}-\frac{1}{\rho}\frac{\partial f}{\partial \theta}\right)\sin 2(\theta-\alpha)}{\left(1+\left(\frac{\partial f}{\partial \rho}\cos(\theta-\alpha)-\frac{1}{\rho}\frac{\partial f}{\partial \theta}\sin(\theta-\alpha)\right)^2\right)\left(1+\left(\frac{\partial f}{\partial \rho}\right)^2+\frac{1}{\rho^2}\left(\frac{\partial f}{\partial \theta}\right)^2\right)^{1/2}}$$

(4.17)

Expression 4.16 can be transformed into:

$$c_\alpha = \frac{\frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}\right)+\left[\frac{1}{4}\left(\frac{\partial^2 f}{\partial x^2}-\frac{\partial^2 f}{\partial y^2}\right)^2+\left(\frac{\partial^2 f}{\partial x \partial y}\right)^2\right]^{1/2}\cos 2(\alpha-\psi)}{\left(1+\left(\frac{\partial f}{\partial x}\cos\alpha+\frac{\partial f}{\partial y}\sin\alpha\right)^2\right)\left(1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2\right)^{1/2}}$$

(4.18)

Which is quite similar to the Euler relation described later in Sec. 4.3 and where ψ is the orientation of the cylindrical axis, given by:

$$\tan\psi = \frac{2\frac{\partial^2 f}{\partial x \partial y}}{\frac{\partial^2 f}{\partial x^2}-\frac{\partial^2 f}{\partial y^2}}$$

(4.19)

4.3 Accuracy of the Approximate Formula to Determine Curvatures in any Direction

We have pointed out that the expression 4.20 provide only an approximate value for the curvatures in directions different from those of the gradient direction and perpendicularly to the gradient. Exact results can be obtained only using the results from the fundamental forms of surfaces, as studied in differential geometry. However, the result is quite accurate. An expression for the curvature error can be calculated by taking the difference between the approximate expression obtained here (Eq. 4.15) and the exact expression (Eq. 4.16) obtained in differential geometry books (Stoker 1969) and, as follows:

$$\Delta c_\alpha = \frac{\frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}\right)+\frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}-\frac{\partial^2 f}{\partial y^2}\right)\cos 2\alpha+\frac{\partial^2 f}{\partial x \partial y}\sin 2\alpha}{\left[1+\left(\frac{\partial f}{\partial x}\cos\alpha+\frac{\partial f}{\partial y}\sin\alpha\right)^2\right]^{3/2}\left[1+\left(\frac{\partial f}{\partial x}\sin\alpha-\frac{\partial f}{\partial y}\cos\alpha\right)^2\right]^{1/2}} - \frac{\frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}\right)+\frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}-\frac{\partial^2 f}{\partial y^2}\right)\cos 2\alpha+\frac{\partial^2 f}{\partial x \partial y}\sin 2\alpha}{\left(1+\left(\frac{\partial f}{\partial x}\cos\alpha+\frac{\partial f}{\partial y}\sin\alpha\right)^2\right)\left(1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2\right)^{1/2}} \tag{4.20}$$

and thus, the error divided by the curvature is:

$$\frac{\Delta c_\theta}{c_\theta} = \frac{\left(\frac{N}{D_1}-\frac{N}{D_2}\right)}{\left(\frac{N}{D_1}\right)} = \left(\frac{D_2-D_1}{D_2}\right) \tag{4.21}$$

Hence, we may easily find:

$$\frac{\Delta c_\alpha}{c_\alpha}=\frac{\left(1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2\right)^{1/2}-\left(1+\left(\frac{\partial f}{\partial x}\cos\alpha+\frac{\partial f}{\partial y}\sin\alpha\right)^2\right)^{1/2}\left(1+\left(\frac{\partial f}{\partial x}\sin\alpha-\frac{\partial f}{\partial y}\cos\alpha\right)^2\right)^{1/2}}{\left(1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2\right)^{1/2}}$$

(4.22)

which, after some algebraic manipulation, and using Eq. 4.22 for the orientation of the gradient, this expression can be transformed into:

$$\frac{\Delta c_\alpha}{c_\alpha}=1-\frac{1}{\left(1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2\right)^{1/2}}\times$$
$$\left[\begin{array}{l}1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2+\frac{1}{8}\left(\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2\right)^2\\ -\frac{1}{8}\left(\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2\right)^2\cos 4\phi\cos 4\alpha\\ -\frac{1}{8}\left(\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2\right)^2\sin 4\phi\sin 4\alpha\end{array}\right]^{1/2}$$

(4.23)

Since this error is quite small, using a Taylor expansion we can find:

$$\frac{\Delta c_\alpha}{c_\alpha} = 1 - \left[1 + \frac{\left|\nabla f(x,y)\right|^4}{8\left(1+\left|\nabla f(x,y)\right|^2\right)} \left(1 - \cos 4\left(\alpha - \phi\right)\right) \right]^{1/2} \tag{4.24}$$

If we add Δc_α from this expression to Eq. 4.15, the small remaining error is compensated, thus obtaining an exact result. As we can observe, the error amplitude depends only on the magnitude of the gradient, or maximum slope of the tangent plane at the point where the curvature is evaluated. We might observe that the error is zero in the directions of maximum and minimum slope (gradient direction) and the direction of zero slope (perpendicularly to the gradient). The maximum error is at ± 45^0 with respect to the gradient direction.

In Fig. 13 we have a polar representation of the approximate and the exact curvatures in sampled directions for a point in a sphere with a radius of curvature equal to 7.722 mm, which is the average radius of the human cornea, at a point at a distance 3.0 mm away from the optical axis and at an angle of 30^0, with the horizontal line. The radial distance to the points is the calculated curvature with the approximate expression (blue dots) and with the exact expression obtained from the fundamental form of surfaces (red dots). The red and blue dots almost overlap, since the difference between these calculated curvatures is smaller than 1%. They exact overlap, since the error is zero in the direction of the gradient and its perpendicular direction, as indicated by the cross.

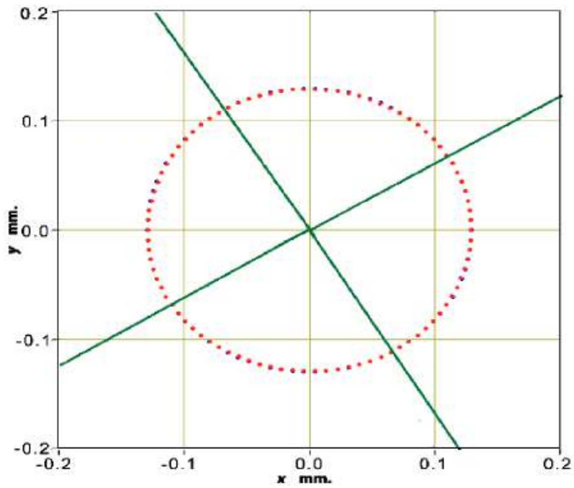


Figure 13. - Polar representation of the approximate and the exact curvatures in sampled directions for a point in a sphere with radius of curvature 7.722 mm, the average radius of the human cornea, at a point at a distance 3.0 mm from the optical axis and at an angle of 30^0, with the horizontal line.

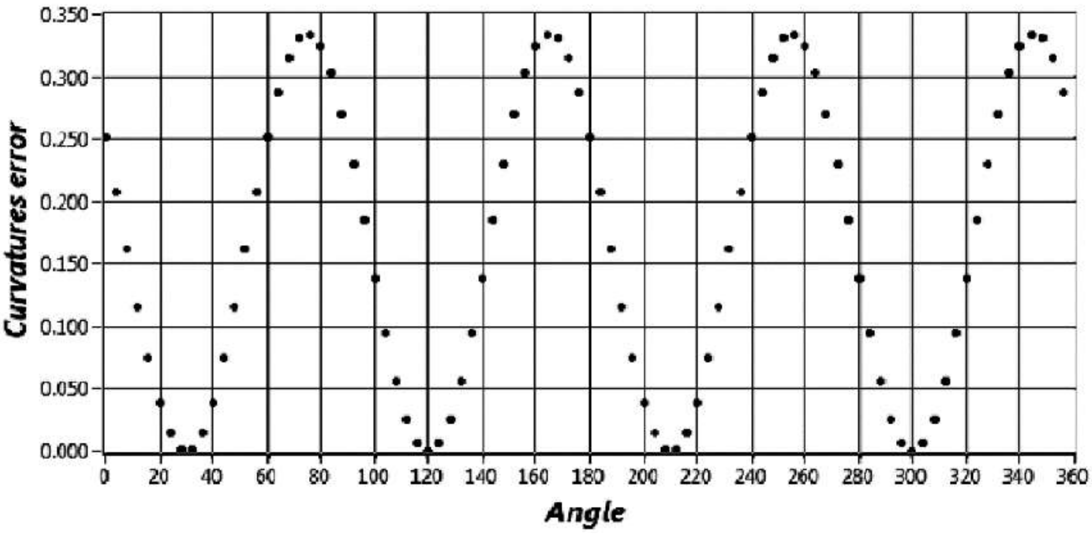


Figure 14. – Error in the calculation of the curvatures in all directions for a point in the sphere.

Figure 14 shows the error calculation of the curvature in all directions for a point in the sphere. Another example is in Figs, 15 and 16. Figure 15 shows a polar representation of the approximate and the exact curvatures in all directions for a point in a sphere-cylindrical surface with a cylindrical curvature 0.04 1/mm at 45^0 and a spherical radius of curvature equal to 7.722 mm, the average radius of the human cornea, at a point at a distance 5.0 mm from the optical axis and at an angle of 30^0, with the horizontal line. Since the radius of curvature of the sphere is 7.7 mm and the off-axis distance for the point where the curvature is measured is near the edge, equal to 5.0 mm the error is much larger than in the preceding example. The cross indicates the direction of the gradient and its perpendicular direction.

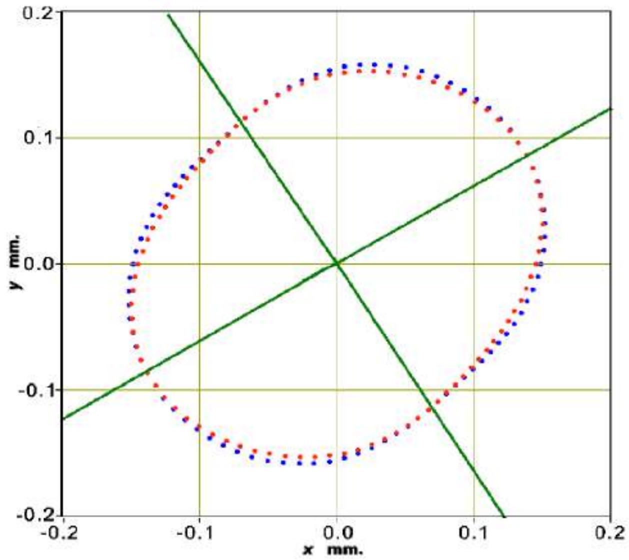


Figure 15. - Polar representation of the approximate and the exact curvatures in all directions for a point in a sphero-cylindrical surface with a cylindrical curvature 0.04 1/mm at 45^0 and a spherical radius of curvature 7.722 mm, the average radius of the human cornea, at a point at a distance 5.0 mm from the optical axis and at an angle of 30^0, with the horizontal line.

Curvature

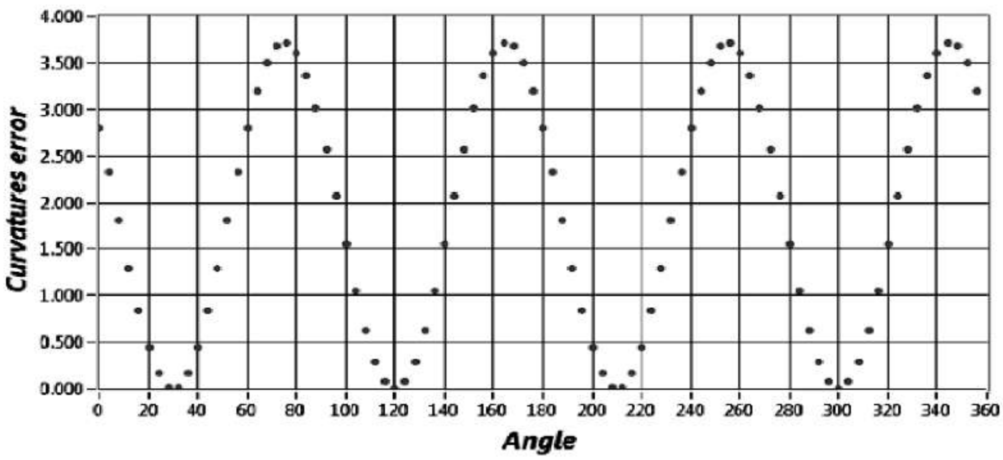


Figure 16. - Error in the calculation of the curvatures in all directions for a point in the sphero-cylindrical surface in Fig. 15

Where the figure 16 shows the calculation of the curvature for a point in the sphero-cylindrical surface.

Curvatures

4.4 Euler Curvature Formula

Generalized Euler´s normal equation for every direction is a useful equation. This is the case for the curvature calculation of a sphere, a cylinder, for the classic ophthalmic curvature specification and to describe a pathological corneal curvature.

The generalization of the Euler normal equation to any direction. It is very useful if we want some classic aspects of the local curvature. The curvature of the sphere and curvature of the cylinder, classic definition in the ophthalmic area, and for pathologic diagnostic in the cornea:

$$\kappa_1 \cos(\alpha)^2 + \kappa_2 \sin(\alpha)^2 = c_N \tag{4.27}$$

$$\kappa_1 \cos(\alpha + \psi)^2 + \kappa_2 \sin(\alpha + \psi)^2 = c_\alpha \tag{4.28}$$

$$\kappa_1 + \left(\kappa_2 - \kappa_1\right) \sin(\alpha + \psi)^2 = c_\alpha \tag{4.29}$$

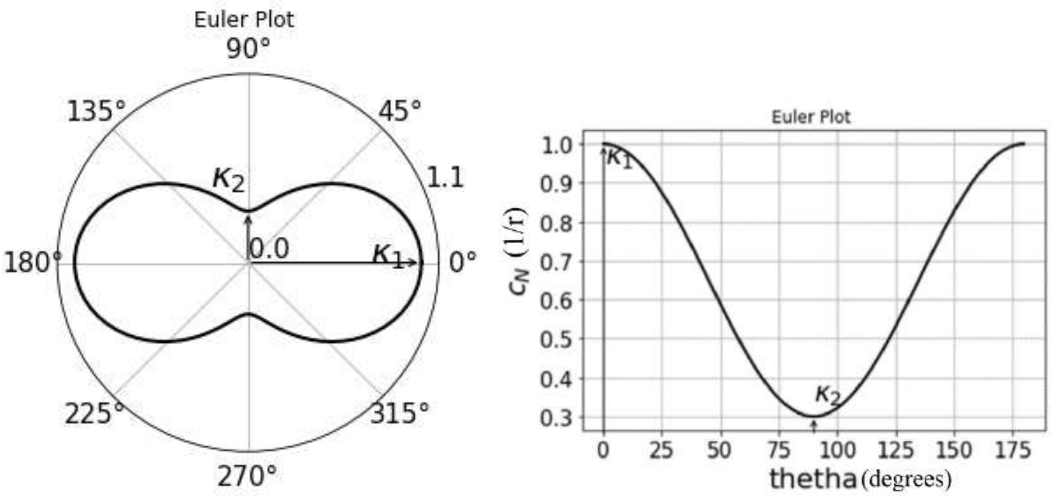
$$\kappa_1 + \left(\kappa_2 - \kappa_1\right)\left(\frac{1 - \cos(2(\alpha + \psi))}{2}\right) = c_\alpha \tag{4.30}$$

$$\left(\kappa_1 + \kappa_2\right)\frac{1}{2} + \left(\kappa_1 - \kappa_2\right)\left(\frac{\cos(2(\alpha + \psi))}{2}\right) = c_\alpha \tag{4.31}$$

Now, we can express the principal curvatures in a most common expression that relates the spherical curvature and the cylindrical curvature, that expressions are very useful in the field of ophthalmology.

$$\left(c_{sph} \right)_{\alpha} + \left(c_{cyl} \right)_{\alpha} \cos(2(\alpha + \psi)) = c_{\alpha} \tag{4.32}$$

In figure 17 we can observe the polar a) and the Cartesian b) distribution of the Euler equation.



a) b)

Figure 17. – a) Polar graph and b) Cartesian graph of principal curvatures with k1 = 1 and k2 = 0.3 with the axes at zero degrees.

In figure 18 we can observe the polar a) and the Cartesian b) plot of the Euler equation. In this particular case, both principal curvatures have the same value than that of the figure 17, however in this case the angle of maximum curvature as respect to the horizontal (x-axis) (ψ) is 45 degrees. Visually the Cassini oval in the polar plane have an angle of the same value and in the case of the Cartesian plane, where the value for the maximum curvature has moved from the origin.

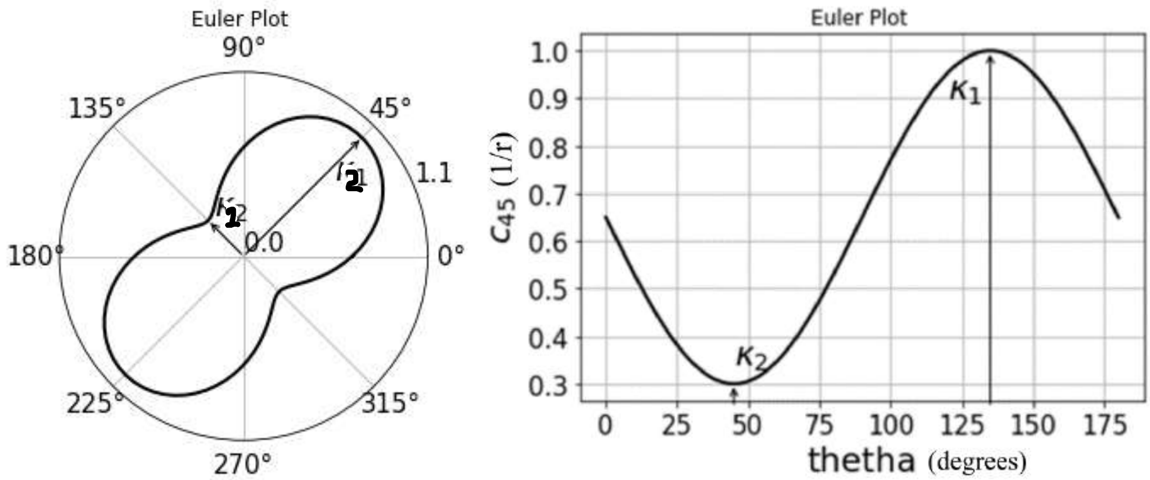


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Figure 18. – a) Polar graph and b) Cartesian graph of principal curvatures with k1 = 1 and K2 = 0.3 with the axes at 45°.

In figure 19 the Euler equation relates the case of a cylinder, in this case k2 is zero.

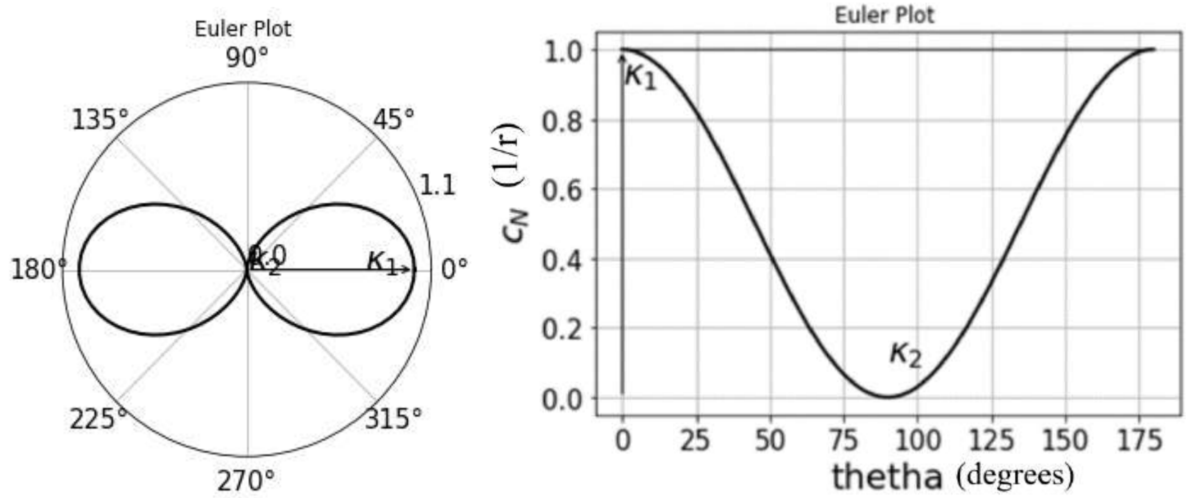


Figure 19. - a) Polar graph and b) Cartesian graph of principal curvatures with k1 = 1 and k2 = 0, with the axes at 0 degrees. The Case of a local Curvature with the shape of a Cylinder.

In figure 20 we can observe the polar a) and the Cartesian b) distribution of the Euler equation. In this particular case, both principal curvature have the same value and describe a circle in the polar plane and a line in the Cartesian plane.

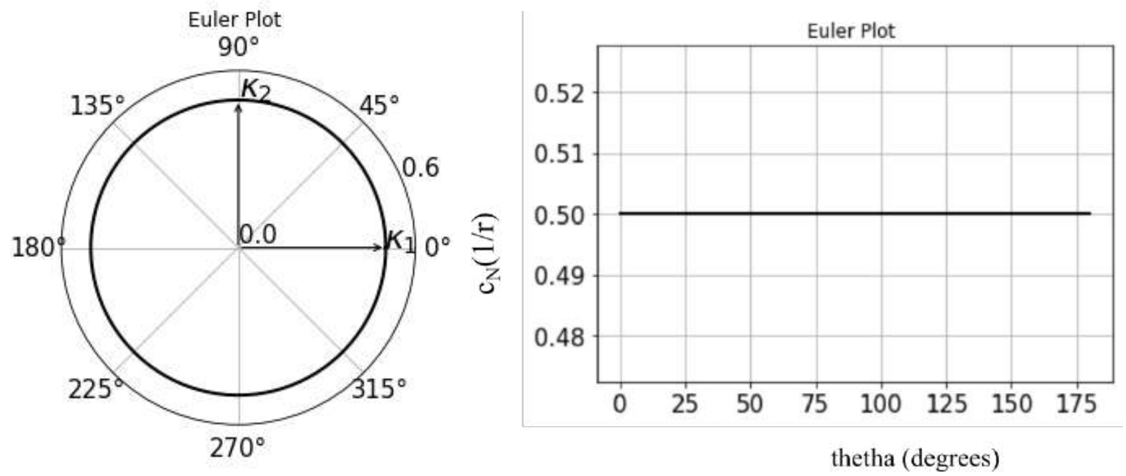


Figure 20. - a) Polar graph and b) Cartesian graph of principal curvatures with k1 = 0.5 and k2 = 0.5, with the axes at 0 degrees. The Case of a local Curvature with the shape of a Sphere.

A point in a surface contains an infinite number of normal planes in all possible directions. These normal planes at a point in a surface in most surfaces and points have different values for different directions, except at points with perfectly spherical or flat shapes.

The expression for the curvature in any direction θ can be expressed in the following form, known as the Euler's curvature formula Eq. 4.32, graphically represented by a closed figure frequently resembling an ellipse, but sometimes it is more like a nut.

The Euler formula representing the polar distribution of the curvature for different angular directions is illustrated in Fig. 21, with an axis orientation $\psi = 0^0$ and different values of the ratio of the cylindrical curvature to the spherical curvature.

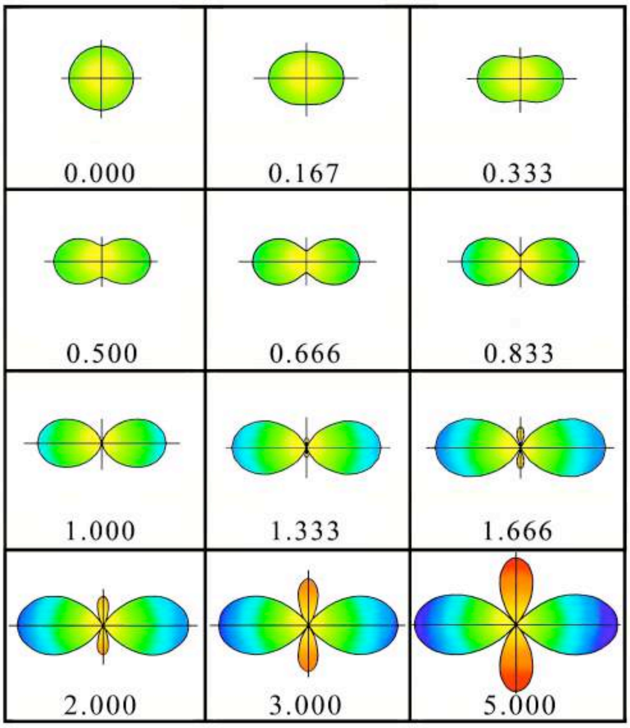


Figure 21. - Polar plots of the Euler formula for an axis orientation ψ equal to 0^0 and different values of the ratio of the cylindrical curvature to the spherical curvature. The color indicates the magnitude of the curvature.

For any kind of surfaces, except that of a sphere, the curvature at any point is not necessarily constant, but it is variable with the direction, in a nearly ellipsoidal manner. In other words, a plot of the normal curvature as a function of the angle describes a closed path, as in Fig. 22, with an inclination ψ of the cylinder axis, given by Eq. 4.33.

There is a maximum value of the curvature κ_1 in one direction and a minimum value κ_2 in an orthogonal direction. These are the two *principal* curvatures. The principal curvatures are the maximum and minimum local curvatures, which are always perpendicular to each other. If the principal curvatures at a point in an optical surface are κ_1 at an angle $\alpha = \psi \pm \pi$ and κ_2 at an angle $\alpha = \psi \pm \pi + \pi/2$ the curvature along a direction θ can also be written in terms of the principal curvatures, as illustrated in Fig. 22

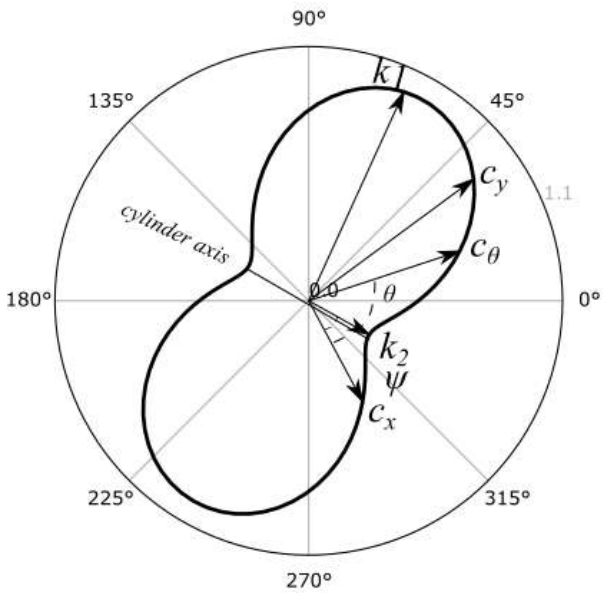


Figure 22. - Variation in the value of the normal curvature c_α for all possible directions. The principal curvatures are orthogonal to each other.

Analytically, the normal curvature can be represented by the Euler curvature formula which is illustrated in Fig. 23(a) as:

$$\begin{aligned} c_\alpha &= \left(\frac{\kappa_1 + \kappa_2}{2}\right) + \left(\frac{\kappa_1 - \kappa_2}{2}\right)\cos 2(\alpha - \psi) \\ &= \left(c_{sph}\right)_a + \left(c_{cyl}\right)_a \cos 2(\alpha - \psi) \end{aligned} \tag{4.33}$$

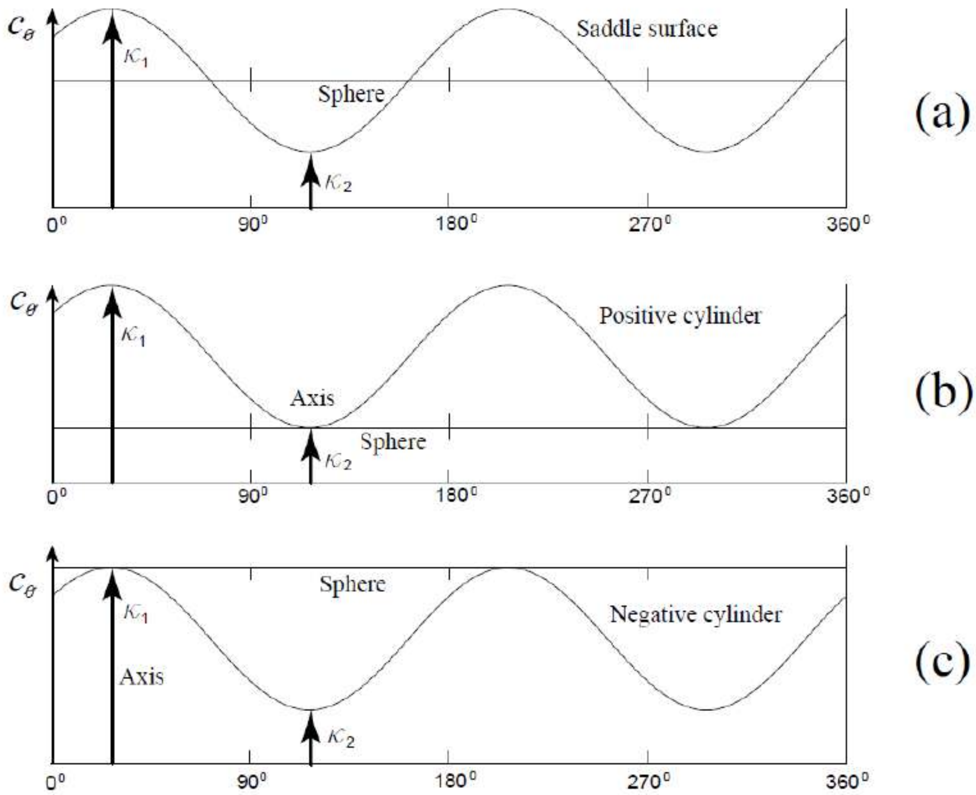


Figure 23. – Curvature variation with the angle at a point in a surface. The total curvature can be expressed as a sum of a spherical and an astigmatic curvature. The spherical component can have a) a curvature equal to the average curvature, b) a curvature equal to its minimum value and c) curvature equal to its maximum value.

Curvature

At the normal plane at an angle ψ the curvature has its maximum value κ_1. In a sphero-cylindrical or toroidal lens the curvature changes in the same manner for planes with different orientations containing the optical axis. If the Euler expression for curvatures is used, κ_1 and κ_2 are the maximum and minimum curvatures, respectively.

The Euler expression for local curvatures can be written in several other different equivalent manners, for example:

$$\begin{aligned} c_\alpha &= \kappa_2 + \left(\kappa_1 - \kappa_2\right)\cos^2(\alpha - \psi) \\ &= \left(c_{sph}\right)_b + \left(c_{cyl}\right)_b \cos^2(\alpha - \psi) \end{aligned} \tag{4.34}$$

as illustrated in Fig. 23(b) and as in Fig. 23(c):

$$\begin{aligned} c_\alpha &= \kappa_1 - \left(\kappa_1 - \kappa_2\right)\cos^2(\alpha - \psi \pm 90^0) \\ &= \left(c_{sph}\right)_c + \left(c_{cyl}\right)_c \cos^2(\alpha - \psi \pm 90^0) \end{aligned} \tag{4.35}$$

where c_{sph} is the spherical curvature and c_{cyl} is the cylindrical curvature. Eqs. 4.36 and 4.37 (Figs. 23(b) and 23(c) are said to be transposed one from the other. A simple method to go from one of them to the other is through three simple steps:

1.- The new first coefficient, (also called the sphere) is obtained by summing the old sphere with the old second coefficient (also called the cylinder). This is the new sphere value.

2.- The new cylinder is obtained by changing the sign of the old cylinder.

3.- The axis orientation ψ is rotated $\pm 90^0$.

This can be written as:

Curvatures

$$\begin{aligned}
\left(c_{sph}\right)_c &= \left(c_{sph}\right)_b + \left(c_{cyl}\right)_b \\
\left(c_{cyl}\right)_c &= -\left(c_{cyl}\right)_b \\
\psi_c &= \psi_b + 90^0
\end{aligned} \tag{4.36}$$

or

$$\begin{aligned}
\left(c_{sph}\right)_b &= \left(c_{sph}\right)_c + \left(c_{cyl}\right)_c \\
\left(c_{cyl}\right)_b &= -\left(c_{cyl}\right)_c \\
\psi_b &= \psi_c + 90^0
\end{aligned} \tag{4.37}$$

Another possible representation of the Euler curvature formula is:

$$c_\alpha = \kappa_1 \cos^2\left(\alpha - \psi\right) + \kappa_2 \sin^2\left(\alpha - \psi\right) \tag{4.38}$$

4.5 Mean, Gaussian and Cylindrical Curvatures

Besides the previously described curvatures, in differential geometry, the mean curvature c_{mean} and the *Gaussian* curvature c_{gauss} have been defined as the arithmetic average and the product, respectively, of the two principal curvatures, as follows:

$$c_{mean} = \frac{\kappa_1 + \kappa_2}{2} \qquad \text{and} \qquad c_{gauss} = \kappa_1 \kappa_2 \tag{4.39}$$

These two curvatures, mainly the mean curvature, had been used to detect some important shape characteristics in the cornea of the human eye, for example, the presence of keratoconus, a deformation and thinning near the center, producing a cone-like bulking of the cornea [20] . Using Euler Eq. 4.28 we may find the mean curvature as the arithmetic average of the two principal curvatures or in a more general manner as the arithmetic average of any two curvatures in orthogonal directions, as follows:

$$c_{mean} = \frac{c_\alpha + c_{\alpha+90^0}}{2} = \frac{\kappa_1 + \kappa_2}{2} = \frac{c_g + c_h}{2} \tag{4.40}$$

Using here Eq. 4.17 we can obtain the mean curvature, as follows:

$$c_{mean} = \frac{\left(\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2}\right)}{2\left(1+\left(\dfrac{\partial f}{\partial x}\cos\alpha + \dfrac{\partial f}{\partial y}\sin\alpha\right)^2\right)\left(1+\left(\dfrac{\partial f}{\partial x}\right)^2+\left(\dfrac{\partial f}{\partial y}\right)^2\right)^{1/2}} \tag{4.41}$$

which, for surface with symmetry of revolution, becomes:

$$c_{mean} = \left(\frac{\dfrac{\partial^2 f}{\partial \rho^2} + \dfrac{1}{\rho}\dfrac{\partial f}{\partial \rho}\left(1+\left(\dfrac{\partial f}{\partial \rho}\right)^2\right)}{2\left[1+\left(\dfrac{\partial f}{\partial \rho}\right)^2\right]^{3/2}}\right) \tag{4.42}$$

If the tangent plane is horizontal, *i.e.*, if the slopes in any direction are zero, the mean curvature can be written as half the Laplacian, as follows:

$$c_{mean} = \frac{1}{2}\nabla^2 f = \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) \tag{4.43}$$

In polar coordinates:

$$c_{mean} = \left(\frac{\dfrac{\partial^2 f}{\partial \rho^2}\left(1+\dfrac{1}{\rho^2}\left(\dfrac{\partial f}{\partial \theta}\right)^2\right)+\dfrac{1}{\rho^2}\dfrac{\partial^2 f}{\partial \theta^2}\left(1+\left(\dfrac{\partial f}{\partial \rho}\right)^2\right)+\dfrac{1}{\rho}\dfrac{\partial f}{\partial \rho}\left(1+\left(\dfrac{\partial f}{\partial \rho}\right)^2\right)}{2\left[1+\left(\dfrac{\partial f}{\partial \rho}\right)^2\right]^{3/2}\left[1+\dfrac{1}{\rho^2}\left(\dfrac{\partial f}{\partial \theta}\right)^2\right]^{3/2}} \right) \tag{4.44}$$

which, for surface with symmetry of revolution, becomes:

$$c_{mean} = \left(\frac{\dfrac{\partial^2 f}{\partial \rho^2}+\dfrac{1}{\rho}\dfrac{\partial f}{\partial \rho}\left(1+\left(\dfrac{\partial f}{\partial \rho}\right)^2\right)}{2\left[1+\left(\dfrac{\partial f}{\partial \rho}\right)^2\right]^{3/2}} \right) \tag{4.45}$$

However, if the slopes (first derivatives) are not zero, this expression is not accurate and may have a significant error (Nasrin *et al*, 2018).

Individually, these two curvatures hide any information about the difference between the two principal curvatures, frequently called the *astigmatism*, and just specify the arithmetic and the geometric average of the two principal curvatures, representing the curvatures of two intermediate reference spheres, tangent at the point under consideration. The mean curvature is represented by the radius of a circle in Fig. 24.

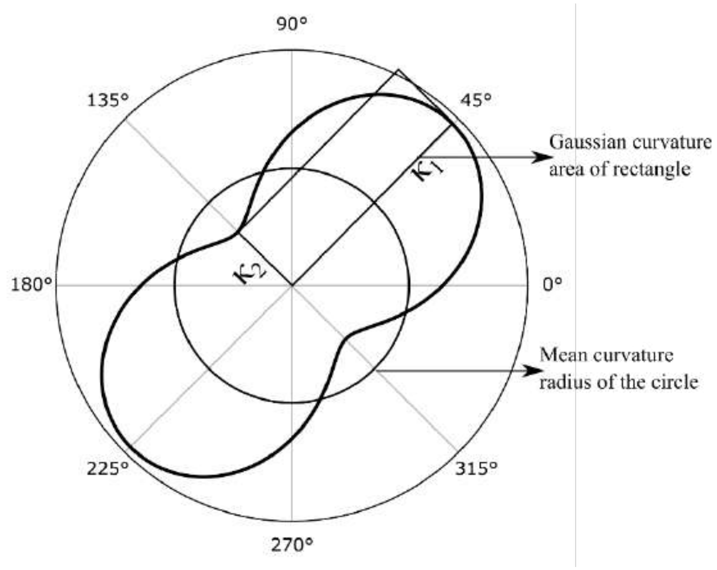


Figure 24. - Polar representation of the Gaussian and mean curvatures.

The Gaussian curvature is the square of the geometric average of the two principal curvatures and has units of $1/mm^2$ instead of 1/mm as the other curvatures. It is the area of the rectangle in the upper part of Fig. 24. A sphere has a constant curvature over the whole surface. Some other surfaces may have a constant Gaussian curvature over the whole surface, for example, Fig. 25 illustrates three surfaces with different constant Gaussian curvatures, inside and outside of the surface. The first surface has a negative value, the second one zero values and the third one positive values.

If both principal curvatures have the same sign, the Gaussian curvature is positive and that point at the surface is said to be an elliptic point. If both principal curvatures are equal, that point at the surface is said to be an *umbilic point* and it is locally spherical. The name comes from latin *umbilicus*, meaning navel. If the two principal curvatures have different signs, the Gaussian curvature is negative and that point at the surface is said to be a hyperbolic or saddle point. If one of the principal curvatures is zero, the Gaussian curvature is zero and that point at the surface is said to be a parabolic point.

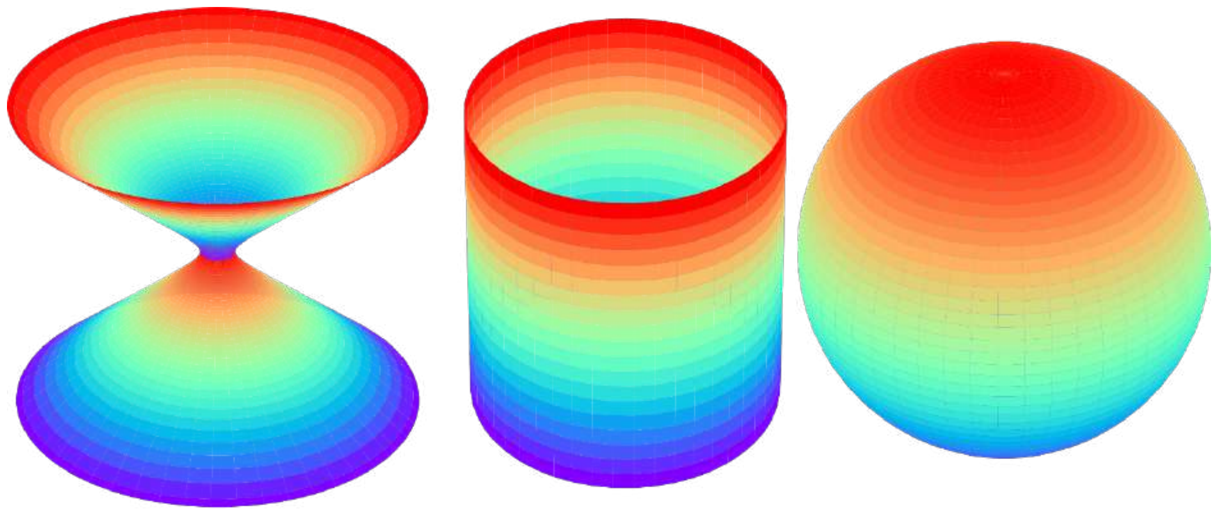


Figure 25. - Three surfaces with different values of the Gaussian curvature. The three surfaces have the same constant value of the Gaussian curvature at all points, inside and outside of the surface.

In general, a surface has different values of the mean and the Gaussian curvatures at different points.

A curvature can be translated into dioptric power in the case of the human eye. The local powers in diopters D_s and D_t are just the curvatures c_s and c_t, multiplied by the index of refraction n= 1.3375 minus one (n -1) as follows:

$$D_s = 0.3375 c_s \quad \text{and} \quad D_t = 0.3375 c_t \tag{4.46}$$

It is important to point out that these dioptric powers are valid for a collimated and narrow beam of light entering perpendicularly to the optical surface at the point where the curvatures are considered. If a wide collimated beam of light enters the cornea of the eye, illuminating the whole pupil, the light will not enter perpendicularly to the surface at all points inside the pupil. Dioptric powers with different definitions may appear in this case. (Roberts 1994). For this reason, frequently, these local powers are said to be paraxial approximations.

The cylindrical curvature, from Eqs. 4.36 And 4.44, is given by:

$$c_{cyl} = \left(\frac{\kappa_1 - \kappa_2}{2}\right) = \frac{\left[\frac{1}{4}\left(\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2}\right)^2 + \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2\right]^{1/2}}{\left(1 + \left(\frac{\partial f}{\partial x}\cos\alpha + \frac{\partial f}{\partial y}\sin\alpha\right)^2\right)\left(1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right)^{1/2}} \tag{4.47}$$

After the mean and the cylindrical local curvatures are calculated with Eqs. 4.31 and 4.45, the principal curvatures can quite easily be obtained.

Using the first (*E, G and F*) and the second fundamental forms (N, L and M) of the differential geometry we can find the definition of the Mean, Gaussian and principal curvatures.

Mean Curvature

$$H = \frac{EN + GL + 2FM}{2(EG - F^2)} = \frac{1}{2}(\kappa_1 + \kappa_2) \tag{4.48}$$

Gaussian Curvature

$$K = \frac{LN - M^2}{EG - F^2} = \kappa_1 \kappa_2 \tag{4.49}$$

Principal Curvatures

$$\kappa_1 = H + \sqrt{H^2 - K}$$

$$\kappa_2 = H - \sqrt{H^2 - K} \tag{4.50}$$

5 Calculating Astigmatic Parameters with Three Measurements

The curvatures along the gradient and perpendicularly to it do not provide all the information about the curvature variation with normal plane orientation. A third parameter is needed in order to obtain the Euler equation and thus the curvatures in all directions. This information also allows us to retrieve the cylinder orientation ψ.

In general, to calculate axis orientation, ψ we take a minimum of three measurements of the curvature in three different directions, as in phase shifting techniques used in optical testing [21]. If we set $\theta_1 = \phi$, $\theta_2 = \phi + 45^0$ and $\theta_3 = \phi + 90^0$ and two of these three measurements will be the curvatures along the gradient and along the perpendicular to the gradient, the third measurement is at 45^0 between them. Thus, using the Euler equation we have:

$$c_g = \left(\frac{\kappa_1 + \kappa_2}{2}\right) + \left(\frac{\kappa_1 - \kappa_2}{2}\right)\cos 2(\phi - \psi) \tag{5.1}$$

$$c_{45} = \left(\frac{\kappa_1 + \kappa_2}{2}\right) - \left(\frac{\kappa_1 - \kappa_2}{2}\right)\sin 2(\phi - \psi) \tag{5.2}$$

$$c_h = \left(\frac{\kappa_1 + \kappa_2}{2}\right) - \left(\frac{\kappa_1 - \kappa_2}{2}\right)\cos 2(\phi - \psi) \tag{5.3}$$

Now, from Eqs. 5.2 And 5.3:

$$\tan 2(\phi - \psi) = -\frac{c_{45} - \left(\frac{\kappa_1 + \kappa_2}{2}\right)}{c_g - \left(\frac{\kappa_1 + \kappa_2}{2}\right)} = \frac{c_g + c_h - 2c_{45}}{c_g - c_h} \tag{5.4}$$

Then, the cylindrical component (difference between the two principal curvatures), as:

$$\left(\frac{\kappa_1 - \kappa_2}{2}\right) = \left[c_g - \left(\frac{\kappa_1 + \kappa_2}{2}\right)\right]\left[1 + \tan^2 2(\phi - \psi)\right]^{1/2} \tag{5.5}$$

but using 5.1 and 5.3:

$$\left(\frac{\kappa_1 - \kappa_2}{2}\right) = \left(\frac{c_g - c_h}{2}\right)\left[1 + \left(\frac{c_g + c_h - 2c_{45}}{c_g - c_h}\right)^2\right]^{1/2} \tag{5.6}$$

The principal curvatures can be obtained from Eqs. 5.2 and 5.4:

$$\kappa_1 = \frac{\left(c_g + c_h\right)}{2} + \frac{\left(c_g - c_h\right)}{2}\left[1 + \left(\frac{c_g + c_h - 2c_{45}}{c_g - c_h}\right)^2\right]^{1/2}$$
$$\kappa_2 = \frac{\left(c_g + c_h\right)}{2} - \frac{\left(c_g - c_h\right)}{2}\left[1 + \left(\frac{c_g + c_h - 2c_{45}}{c_g - c_h}\right)^2\right]^{1/2} \tag{5.7}$$

5.1 Sagittal and Tangential Curvatures for Conic Surfaces

Conic surfaces are a particular case of surfaces with rotational symmetry. For conic surfaces, the sagittal, azimuthal or axial curvature is given by [22]:

$$c_s = \frac{c}{\left[1 - Kc^2S^2\right]^{1/2}} \tag{5.8}$$

and the tangential, radial or instantaneous curvature is given by:

$$c_t = \frac{c}{\left[1 - Kc^2S^2\right]^{3/2}} \tag{5.9}$$

As in any rotationally symmetric surface, the two curvatures, sagittal and tangential, are related to each other. In this case:

$$c_s^3 = c^2 c_t \tag{5.10}$$

where c is the vertex curvature, *ie*., the curvature at the intersection of the optical surface with its optical axis. The local radius of curvature is equal to the radius of curvature r plus the aberration of the normals, represented by Δr, as illustrated in Fig. 26. Since the slope of the line going from the point **P** to the local center of curvature is equal to the first derivative or slope of the aspherical surface, the aberration of the normals can be obtained as:

$$\Delta r = \frac{S}{\left(\dfrac{\mathrm{d}z}{\mathrm{d}S}\right)} + z - r \tag{5.11}$$

which, for conic surfaces becomes:

$$\Delta r = -Kz \tag{5.12}$$

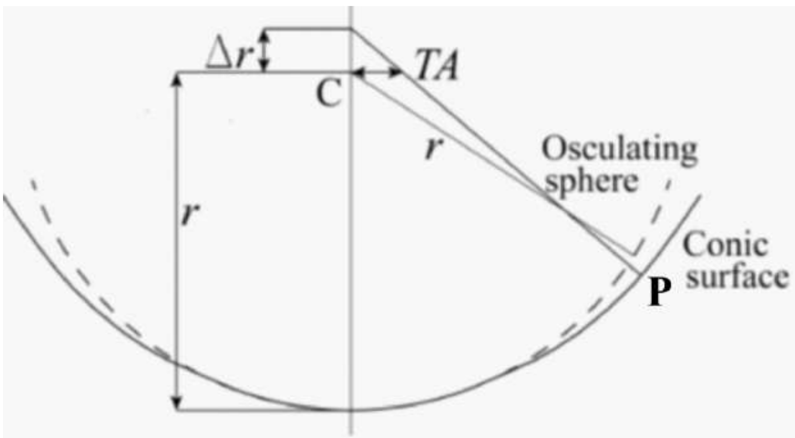


Figure 26. - A conic surface with its osculating sphere, illustrating the transverse aberration.

The anterior surface of the cornea is frequently assumed to be ellipsoidal, with or without rotational symmetry [23]. With this fact in mind Harris (2006) and later Bektas (2016) studied the curvature of general ellipsoids using the fundamental form of surfaces [24]–[26]. A detailed study of the shape of the cornea and its local curvatures has also been reported by Griffits, Plociniczak and Schliesser (2016a and 2016b).

5.2 Tangential and Sagittal Curvatures for Human Eye Corneas

For a circular optical surface or a wavefront coming out from a circular exit pupil in an optical system, for example, in the human eye corneas, two curvatures are frequently used. One is in the radial direction, the *tangential* or *radial* curvature also called *instantaneous* curvature. The other curvature in a perpendicular direction, the *sagittal* or *azimuthal* curvature c_s, also called *axial* curvature by optometrists and ophthalmologists. (See Fig. 27).

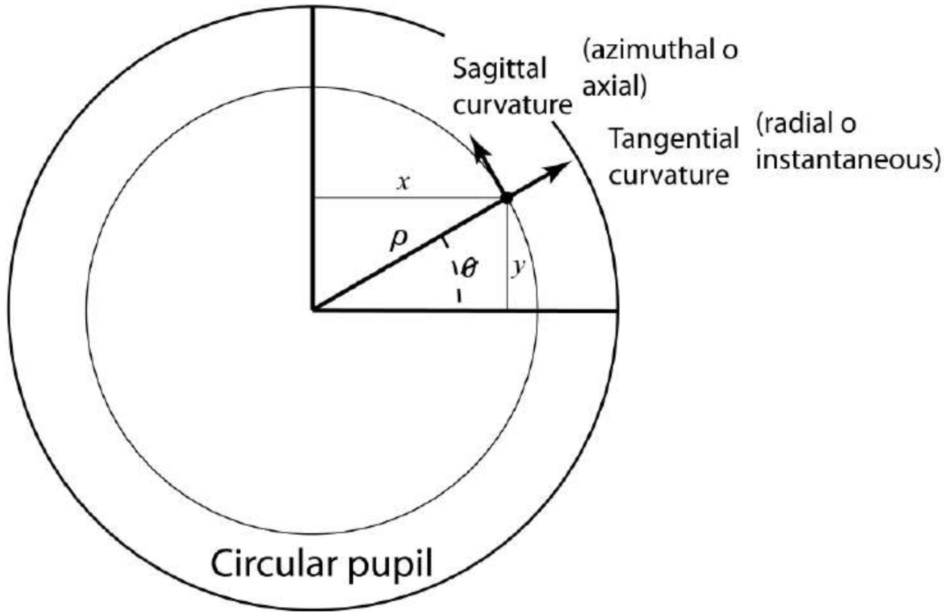


Figure 27. - Tangential or instantaneous and sagittal or axial curvatures in a circular pupil.

Using the general expression in polar coordinates, Eq. 4.16 for local curvatures, the tangential (also called radial or instantaneous) curvatures can be found by setting $\alpha = \theta$, obtaining:

$$c_T = \frac{\dfrac{\partial^2 f}{\partial \rho^2}}{\left(1+\left(\dfrac{\partial f}{\partial \rho}\right)^2\right)\left(1+\left(\dfrac{\partial f}{\partial \rho}\right)^2+\dfrac{1}{\rho^2}\left(\dfrac{\partial f}{\partial \theta}\right)^2\right)^{1/2}} \tag{5.13}$$

and using the same general expression, the sagittal (also called axial or azimuthal) can be obtained by setting $\alpha = \theta + 90^0$:

$$c_S = \frac{\left(\dfrac{1}{\rho}\dfrac{\partial f}{\partial \rho} + \dfrac{1}{\rho^2}\dfrac{\partial^2 f}{\partial \theta^2}\right)}{\left(1+\dfrac{1}{\rho^2}\left(\dfrac{\partial f}{\partial \theta}\right)^2\right)\left(1+\left(\dfrac{\partial f}{\partial \rho}\right)^2+\dfrac{1}{\rho^2}\left(\dfrac{\partial f}{\partial \theta}\right)^2\right)^{1/2}} \tag{5.14}$$

These two expressions are valid for optical surfaces with any aberrations, including those with extreme asymmetries. However, many ophthalmic or optometric surfaces, for example, most human eye corneas, are nearly rotational symmetric, if the aberrations are not very large. In this case, the tangential and sagittal expressions for the local curvatures can be simplified by setting the first and second derivatives of *f* with respect to θ equal to zero, as follows:

$$c_T = \frac{\dfrac{\partial^2 f}{\partial \rho^2}}{\left(1+\left(\dfrac{\partial f}{\partial \rho}\right)^2\right)^{3/2}} \tag{5.15}$$

and:

$$c_S = \frac{\dfrac{1}{\rho}\dfrac{\partial f}{\partial \rho}}{\left(1+\left(\dfrac{\partial f}{\partial \rho}\right)^2\right)^{1/2}} \tag{5.16}$$

Let us now study with more detail these curvatures for surfaces with nearly rotational symmetry. For these surfaces, both curvatures, tangential and sagittal are constant for all values of θ and a given value of ρ. In other words, the sagittal and the rotational maps are rotationally invariant. Now, observing Fig. 28, we see that the axial (or sagittal) curvature c_S, is the curvature at a point on the intersection of the optical surface with a plane passing through the local center of curvature, which is perpendicular to the plane containing the optical axis (a tangential plane). Thus, when the optical surface has rotational symmetry, the sagittal (or axial) radius of curvature can be calculated by tracing a ray passing through the point **P** and measuring its distance to the intersection with the optical axis. This is the reason for the name "axial curvature". If the optical surface does not have rotational symmetry, the ray passing through the point **P** does not cross the optical axis. This effect is called *skew ray error* in the optometric literature (Klein 1997).

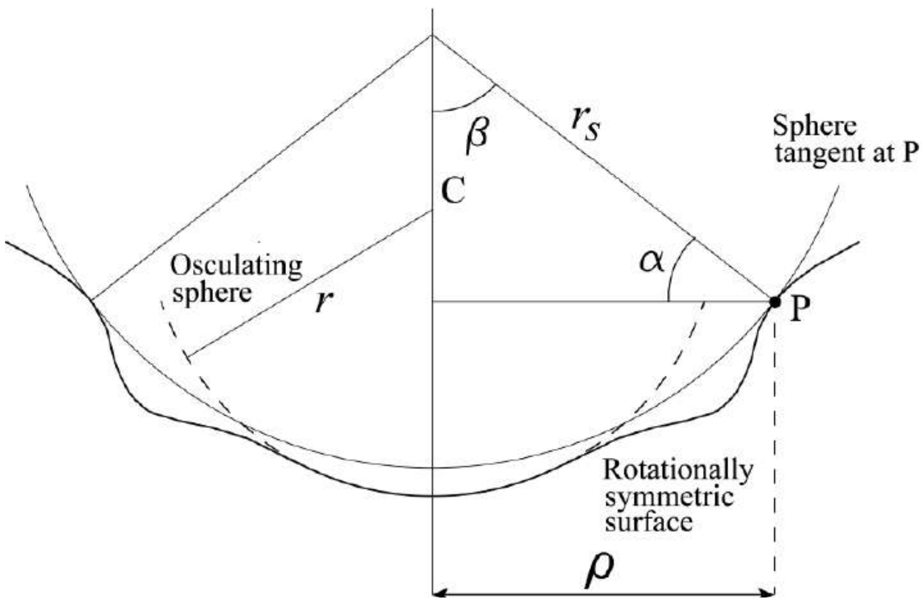


Figure 28. - An optical surface with rotational symmetry, showing the osculating sphere and also a sphere touching the aspherical surface along a ring passing through the point **P**.

When the surface has rotational symmetry about the optical axis, these tangential and sagittal curvatures can be calculated with simpler formulas. In Fig. 28. we have a sphere tangent to a surface with rotational symmetry about the optical axis. The point of tangency is at the point **P** and along a circle containing the point **P**, concentric with the optical surface. Since the sphere and the surface are tangents along this circle, a line being perpendicular to the optical surface is also a radius for the sphere with the axial curvature $1/r_s$.

$$c_S = \frac{1}{r_s} = \frac{\sin\beta}{\rho} = \frac{\tan\beta}{\rho\left[1+\left(\tan\beta\right)^2\right]^{1/2}} \tag{5.17}$$

and using the theorem of Meusnier in Eq. 5.8:

$$c_S = \frac{1}{r_s} = \frac{\cos\alpha}{\rho} = \frac{1}{\rho\left[1+\left(\tan\alpha\right)^2\right]^{1/2}} \tag{5.18}$$

These expressions are exact only for rotationally (axially) symmetric optical surfaces, without any non-rotationally symmetric aberrations [27].

Barbero (2015) has described the concepts of geodesic curvature and geodesic torsion as a metric of the difference between a rotationally symmetric and a non-rotationally symmetric surface[28]. When non-rotational symmetric aberrations are present and this expression is used, important errors appear, mainly at the periphery of the corneal surface.

In the case of surfaces with symmetry of revolution, these are the *tangential* and the *sagittal* curvatures. For surfaces without symmetry of revolution, like the sphero-cylindrical surfaces or astigmatic corneas, in general, the principal curvatures are not in the tangential and sagittal directions. Thus, these principal curvatures are not the same as the averages of the tangential and sagittal curvatures, unless the surface or wavefront deformations are rotationally symmetric.

In surfaces without rotational symmetry the two curvatures, sagittal and tangential are completely independent of each other. Under these conditions it is not possible to derive one type of curvature from the other. There is a lot of confusion in the literature about this topic. However, for surfaces with rotational symmetry, these two curvatures are related to each other by [1], [8, p. 45], [20]:

$$c_t = \frac{\mathrm{d}(\rho c_s(\rho))}{\mathrm{d}\rho} \qquad \text{or} \qquad c_s(\rho) = \frac{1}{\rho}\int_0^S c_t(\rho')\,\mathrm{d}\rho' \tag{5.19}$$

These expressions are also valid for aberrated surfaces as long as the aberrations are rotationally symmetric, or the non-rotationally symmetric aberrations are extremely small compared with the rotationally symmetric component. The local astigmatism axis has its axis is along with the tangential or sagittal directions only if the surface is rotationally symmetric.

When the surfaces do not have rotational symmetry, the two principal curvatures are not the same as the tangential and sagittal curvatures, at any point in the optical surface, which is the case when: a) the surface has a cylindrical or sphero-cylindrical shape, b) when the aberration surface has non-rotationally symmetric deformations.

In general, the principal curvatures cannot be calculated from the sagittal and tangential curvatures only. An extra parameter has to be determined at all points in the optical surface, for example, the orientation of the principal curvatures (cylinder axis) of the difference of their magnitudes, (cylinder magnitude), or curvature in another direction. This means that the sagittal and tangential maps do not provide the whole information about the shape of the surface deformations nor about the curvatures.

Examples of maps for tangential and sagittal curvatures, obtained by computer generation for an astigmatic (sphero-cylindrical) wavefront are shown in Fig. 29.

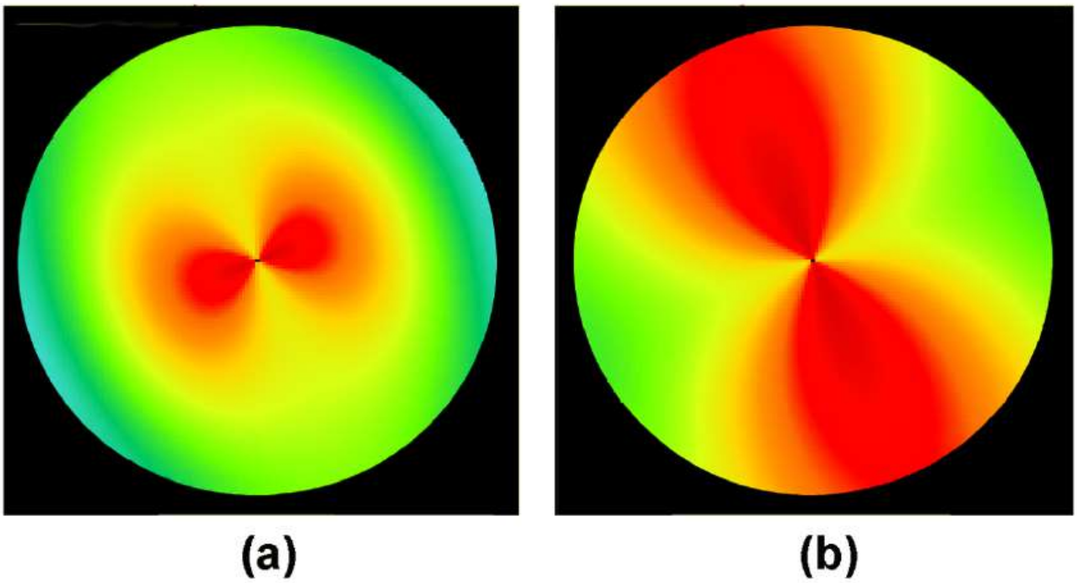


Figure 29.- An example of a tangential map of curvatures (a) and a sagittal map of curvatures (b) for a sphero-cylindrical optical surface or wavefront.

5.3 New Proposed Map of Curvatures

In order to have information about the three parameters, at least two orthogonal curvatures and the cylindrical axis orientation, the circular pupil can be divided into cells, as in Fig. 30, by means of set of circles. As an example, let us consider an array of concentric circles with the points over the circles as in Fig. 31d, with sampling spots in concentric rings. Inside of each cell a plot of the Euler equation (Eq 4.27) is located. At the center of each cell the curvature along several directions is calculated to plot the Euler equation (Eq. 4.27), resembling an ellipse or a nut. Then, the next step is to calculate the maximum and the minimum values ($\kappa 1$ and $\kappa 2$) of the length of the Euler figure for the cells in the whole pupil. In any kind of surface, except in a sphere, the curvature at any point is not necessarily constant, but variable with the direction, in a nearly ellipsoidal shape, known as the Euler formula (Eq. 4.27) resemble or Cassini oval. In other words, a plot of the curvature as a function of the angle describes a closed path, as in Fig. 30, with an inclination ψ of the cylinder axis.

The Euler equation can be plotted by calculating its shape with the curvatures in several different directions from cero degrees to one hundred and eighty degrees. Only half of the circle needs to be calculated due to the symmetry of the curve.

Curvature

The point where the local curvatures are calculated and plotted are in an array similar to the one sometimes used in the Hartmann test, with the sampling points in concentric circles. The cells are nearly hexagonal to the outer of the pupil due to our algorithm. The number of sampling points in each circle is equal to six times the ring number, where the first point is at the *y* axis, that is, it has an angle θ equal to 90°. Thus, any sampling point has the following polar coordinates, as in Fig. 30:

$$\rho = js$$
$$\theta = 90^\circ + \frac{m}{j} 60^\circ \tag{5.20}$$

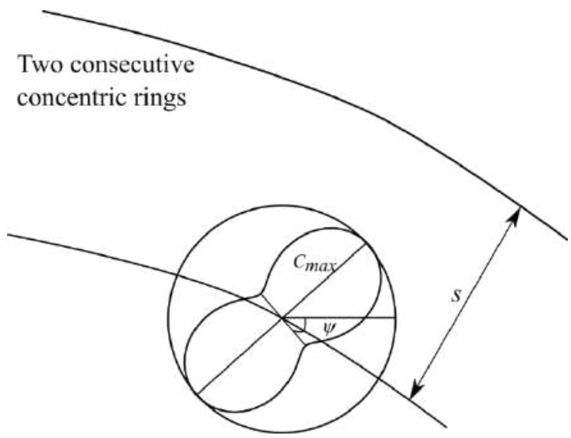
where *j* is the ring number, *m* is the sampling spot number in that ring and s is the rings radial separation, given by:

$$s = \frac{D}{2N+1} \tag{5.21}$$

with N equal to the total number of rings and *D* is the pupil diameter. The Cartesian coordinates x and y for each of these points are given by:

$$x = \rho \cos\theta$$

$$y = \rho \sin\theta$$



Curvatures

Figure 30.- The curvatures represented by the Euler equation (Eq. 4.38), between two consecutive concentric rings in the pupil.

These maps are constructed with the following characteristics:

The Cassini figures are scaled with a factor such that the maximum value of the curvature is smaller than half the separation between two consecutive circles. The smallest Euler figure, with the minimum curvature becomes very small.

b) If color coding is employed, the Cassini Ovals are either red, for positive values of the curvature or blue for negative values of the curvature.

The maximum curvature at the pupil is represented by c_{max} and the minimum curvature by c_{min}. Each Cassini figure at each hexagonal cell has a maximum value (κ1) and a minimum value (κ2), called principal curvatures. Thus, cmax is the largest principal curvature κ1 over the aperture and c_{min} is the smallest principal curvature κ2 over the aperture.

The main advantage of these maps to be described here is that the three parameters, local curvature values, local astigmatism values and axis orientation are illustrated in a single figure.

In Fig. 31we have five examples of wavefront deformations due to primary aberrations and their representation in maps, with the sampling points in concentric rings over the aperture. These maps had been generated with the aberration added to a spherical surface with a radius of curvature close to that of the human cornea (r = 6.7 mm). The curvatures in the primary aberrations component have both, negative (concave) and positive (convex) values, but with magnitudes smaller than the curvature in the sphere. Thus, the local curvatures in all of these maps is always of the same sign (positive) and convex. There are three columns of figures. The first one has elevation maps, represented by Twyman-Green interferograms. The elevation in these interferograms is the difference with respect to the non-aberrated sphere. The second column has the corresponding isometric elevation maps. Finally, in the third column we have the local curvature maps as proposed here. These patterns are plotted calculating only the first and second derivatives from Zernike wavefront or surface descriptions in Eq. 3 for an array of points in a circular pupil.

The map in the first row (Fig. 31a) is for a defocused sphere, with a curvature slightly larger (more convex) than in the original sphere. The local curvatures are the same in all directions and at all sampling points over the aperture. Thus the map has circles, all with the same diameter. Their size is directly proportional to the defocusing magnitude. It important to point out that any other non-spherical surface, even if it has rotational symmetry like, for example in the case of spherical aberration, (Fig. 31d) has some places where there is local astigmatism.

The map in Fig. 31b is for a sphero-cylindrical aberrated surface with almost a cylindrical shape. This aberration is slightly convex along the y direction and strongly concave along the x direction. In the local curvature map the whole aperture is covered with identical ellipses with is maximum size (greatest convex curvature) along the y direction.

Figure 31c has a map for an aberrated surface with a saddle shape, corresponding to astigmatism with its reference sphere close to the aberrated surface. The greatest curvature is in the direction of the convex axis in the y direction. The local curvature map is formed by identical Cassini ovals with its narrow waist (smaller curvature) in the x direction and its larger size (greater curvature) in the y direction. Thus, a table with the two principal curvatures and the astigmatism axis orientation in every sample point could be calculated, and they can roughly estimate from the shape of the plotted Cassini ovals.

In Fig. 31d we have the results for spherical aberration. This is a surface deformation with rotational symmetry. We may observe that the aspherical deformation introduces some astigmatism, but the astigmatic axis is always in the radial or tangential direction. The average size of the Cassini figures changes with their radial position, indicating that the average curvature also changes with the radial position. For an aspheric surface with rotation symmetry, that is, with only spherical aberration the result will be the same.

The last row in Fig. 31e is for the aberration of coma, along the y direction. The upper part of this aberration is concave and the lower part is convex. Thus, the greater local curvatures are in the lower part (y negative) of the aperture and the smallest local curvatures are in the upper part (y negative) of the aperture as shown in the local curvatures map.

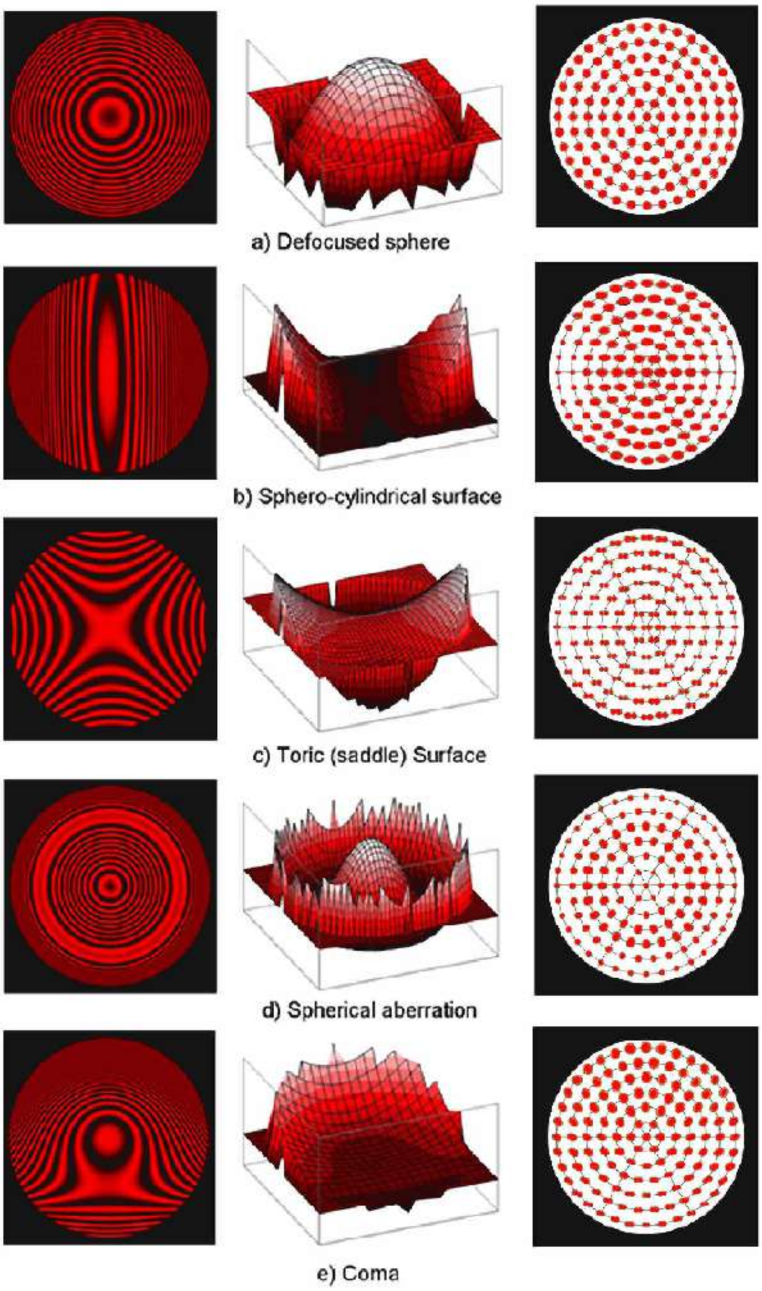
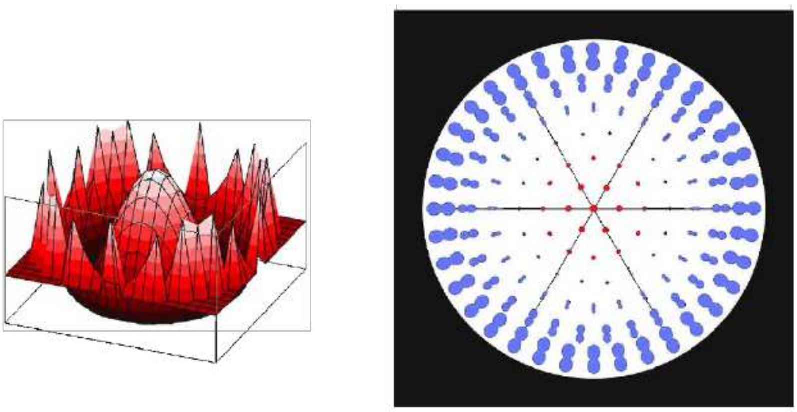
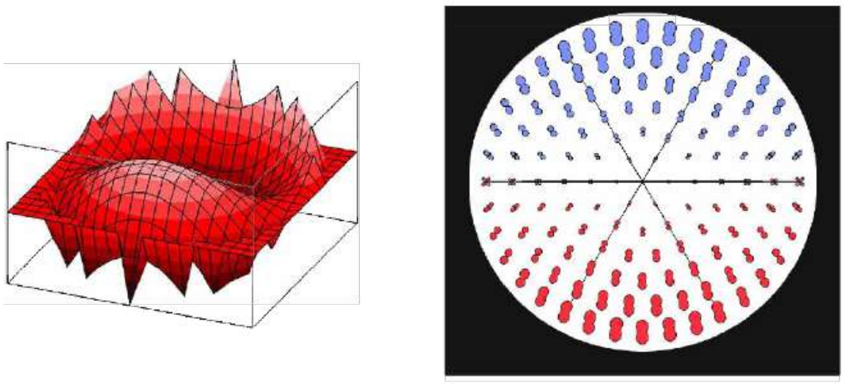


Figure 31.- Five examples of wavefront deformations due to primary aberrations and their representation in maps with sampling points in concentric circles. In these figures, we have six rings. In the first column we can see the Twyman-Green interferograms. In the second column we have the isometric plots of these wavefronts and in the third column we have the proposed maps with the origin in the center of the pupil.

Maps with colors to represent the magnitudes of the curvature are not always necessary. The magnitudes are represented by the size of the Cassini ovoid and its sign is always positive if the aberrations are much smaller than the non-aberrated sphere sagitta. Then, the curvatures are close to the curvature of the sphere. Color codding would be useful only if the curvature range includes both convex and concave curvatures or positive and negative powers inside the aperture as in ophthalmic progressive lenses. This is the case when the aberrations are measured with respect to a reference sphere with a curvature close to that of the average curvature of the surface or wavefront aberration being measured, for example, as previously described in Sec. 4. Then, it is convenient to use red color for the positive curvatures and blue color for the negative curvatures, as illustrated in Fig. 32.



a) Spherical Aberration



b) Coma along y-axis

Figure 32.- Two examples of wavefront deformations due to primary aberrations and their representation in map with sampling points distributed in concentric circles. In the first column

we have the isometric plots of these wavefronts and in the second column we have the proposed maps with the origin in the center of the pupil. These aberrations are measured with respect to the non-aberrated sphere.

5.4 Measurement of Local Curvatures

From the exact expression in Eq. 4.16 we see that the local curvatures at any point in a surface can be calculated if the two first derivatives, the second derivative respect to *x* and *y* are known. So, the first step is to measure them.

Many experimental methods are carried out by ray tracing, nearly always using one of the Hartmann optical arrangements as illustrated in Fig. 33 [29]. Two common Hartmann type configurations are illustrated in Fig. 33. The optical system illustrated in Fig. 33 (a) is used to test a concave optical surface or a convergent wavefront, typically telescope mirrors. Fig. 33 (b) is used mostly used to measure the convex cornea of the human eye. In this system, the virtual images of the light sources are formed in a slightly curved surface behind the corneal surface. If the ovoidal surface containing the light sources is appropriately elongated, the images are in a plane. Frequently, a compromise is taken so that the ovoidal surface is not too elongated and the virtual image surface is not too curved. The camera forms at its focal plane a real image of the light sources to measure the transverse aberrations.

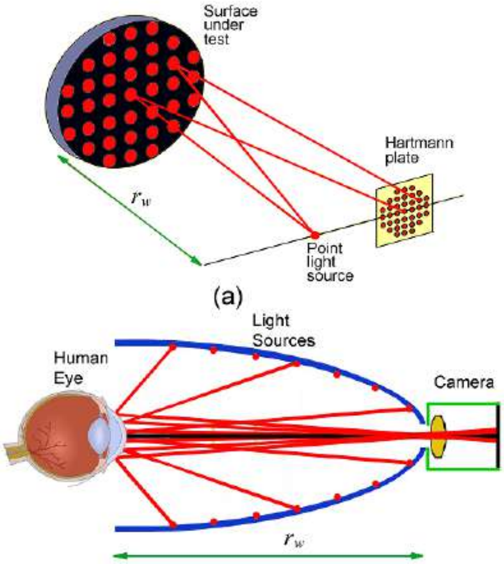


Figure 33. - Two common Hartmann configurations. (a) is used to test a concave optical surface or a convergent wavefront. (b) is used to measure the convex cornea of the human eye.

In both systems the array of light sources can have several possible array configurations, as illustrated in Fig. 34.

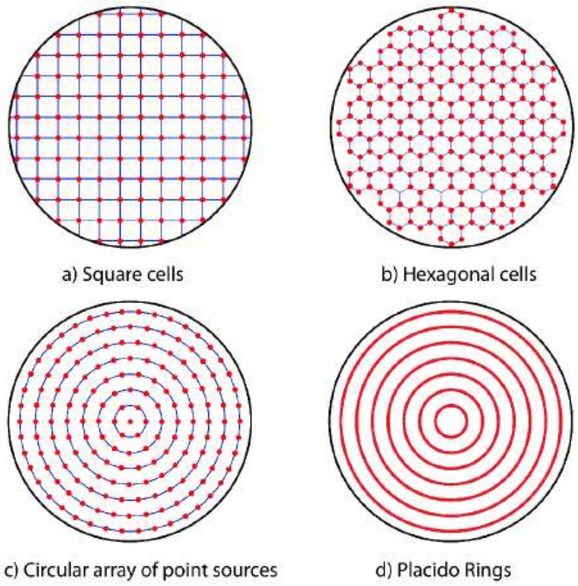


Figure 34. - Several possible arrays of light sources for Hartmann type measuring systems.

In astronomical optical surfaces testing the most common array is with square cells, as in Fig. 34 (a). In ophthalmic instruments, like corneal topographers, the most common array of the light source is with Placido rings in Fig. 34 (d). However, all these light arrays can be used with any system. The simplest one for curvature measurements is the array of squares.

The transverse aberrations are directly proportional to the slopes of the optical surface or wavefront in a square cell, as shown in Fig. 37that is a particular case in a dot. These aberrations are determined by the angular separation between the actual reflected or refracted ray and the path for the ideal ray. This ideal path for the reflected path is calculated for an ideal reference surface. If this reference surface is flat the angle between the actual ray and the ideal ray is twice the slope of the surface at the point where the ray is reflected. However, in most optical arrangements, like those in Fig. 33, the reference surface used to find the reference points for the measurement of the transverse aberrations is a perfectly spherical surface, close to the aberrated real surface. Thus, the measured transverse aberrations are produced by the separation between the aberrated surface and the spherical reference surface, measured perpendicularly to this reference surface.

At this point it is convenient to define the *absolute aberrations* as those of the aberrated surface measured with respect to the reference sphere in a direction parallel to the optical axis. On the other hand, the *relative aberrations* are defined as those of the aberrated surface measured with respect to the reference sphere, perpendicularly to this sphere. Figure 35 illustrates these concepts of the *absolute* z_a and the *relative* z_r. They are related by:

$$z_r = z_a \cos\theta \qquad (5.22)$$

where θ is the angle between their normal to the sphere and the normal to the aberrated sphere. If the aberrated wavefront is almost flat, the reference sphere is a plane and θ is zero.

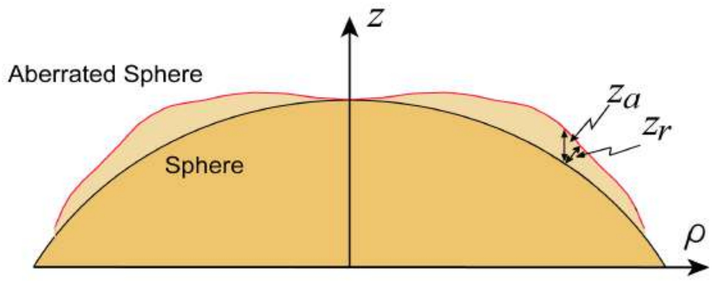


Figure 35.- Illustration of the absolute sagitta, measured with respect to the reference sphere, in a direction parallel to the optical axis and the relative sagitta, measured with respect to the reference sphere and in a direction towards the center of the sphere.

We can change the curvature of the reference sphere of a convex (divergent) wavefront by means of a collimating lens. By use of the Fermat principle we can see that the aberrated wavefront sagitta and its curvatures do not change, as shown in Fig. 36, where the z_{r1} in the divergent wavefront is equal to z_{a2} in the collimated wavefront.

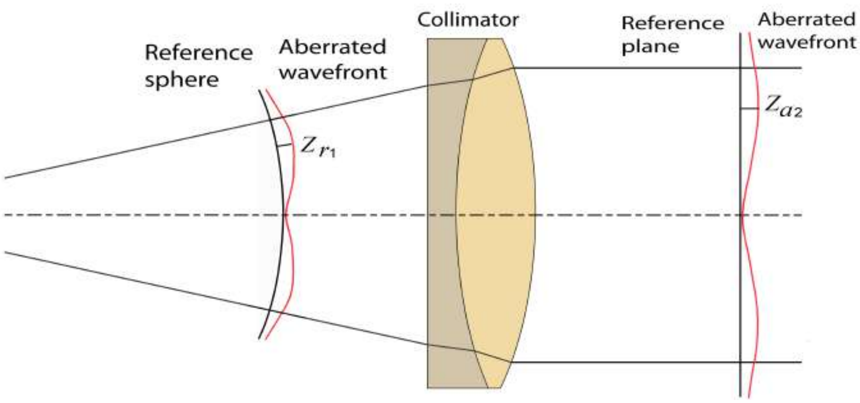


Figure 36. - Flattening of the curved aberrated wavefront by means of a collimator.

In lens design and evaluation programs, the optical surface shape (sagitta distribution) is defined by:

$$z(\rho, \theta) = \frac{c\rho^2}{1+\left[1-(K+1)c^2\rho^2\right]^{1/2}} + \sum_{j=1}^{4} A_n \rho^{2(n+1)} + \sum_{n=1}^{N}\sum_{m=1}^{M} B_{nm} Z_n^m(\rho,\theta) \tag{5.23}$$

where the first term is a conic surface (sphere, ellipsoid or hyperboloid), K is the conic constant, $\rho^2 = x^2 + y^2$, c is the vertex curvature of the reference sphere, A_n are the deformation coefficients, and the last term is a linear combination of Zernike polynomials Z_n^m. It is important to point out that the contributions of these three sagitta terms are all in the direction of the optical axis and not perpendicular to the reference sphere. A possible application of this is for the evaluation of human eye models. The absolute local curvatures can be calculated by obtaining the derivative from this expression and substituting it in Eq. 4.16.

The measured transverse aberrations (Fig. 37) could be quite small if the relative surface deformations are small. Then, an interesting consequence is that the difference between the first derivatives is small and the denominator in the exact curvature expression becomes almost one. In other words, the exact expression for the curvatures is not necessary.

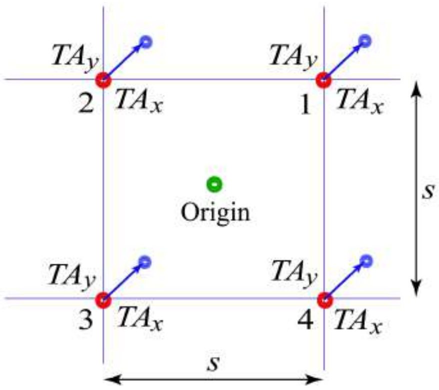


Figure 37.- Square cell with four points where the transverse aberrations are measured.

In this case the derivatives may be obtained by first obtaining the wavefront deformations in each cell in a procedure called zonal method. The reference optical surface deformations $z_r(x, y)$ over the square cell can be represented by:

Curvature

$$
\begin{aligned}
z_r(x,y) &= A_1x + A_2y + A_3(x^2+y^2) + A_4(x^2-y^2) + 2A_5xy \\
&= A_1\rho\cos\alpha + A_2\rho\sin\alpha + A_3\rho^2 + A_4\rho^2\left(\cos^2\alpha - \sin^2\alpha\right) + 2A_5\rho^2\cos\alpha\sin\alpha \\
&= A_1\rho\cos\alpha + A_2\rho\sin\alpha + A_3\rho^2 + A_4\rho^2\cos2\alpha + A_5\rho^2\sin2\alpha \\
&= A_1\rho\cos\alpha + A_2\rho\sin\alpha + A_3\rho^2 + B\rho^2\cos2\psi\cos2\alpha + B\rho^2\sin2\psi\sin2\alpha
\end{aligned}
$$

(5.24)

or, after some algebraic manipulation:

$$z_r = A_1\rho\cos\alpha + A_2\rho\sin\alpha + A_3\rho^2 + \left(A_4^2 + A_5^2\right)^{1/2}\rho^2\cos2\left(\alpha-\psi\right) \tag{5.25}$$

If the aberrations are small, the second derivative in the direction α will give the local curvature in that direction. The astigmatic axis orientation is given by:

$$\tan2\psi = \frac{A_5}{A_4} \tag{5.26}$$

The transverse aberrations and the slopes are related by:

$$
\begin{aligned}
\frac{\partial z_r(x,y)}{\partial x} &= -\frac{2}{r_w}TA_x \\
\frac{\partial z_r(x,y)}{\partial y} &= -\frac{2}{r_w}TA_y
\end{aligned} \tag{5.27}
$$

where r_w is the distance from the observation plane to the screen with the cells or the array of virtual mages an TA_x ad TA_y are the distances from the actual reflected ray and the reference ray when they cross the observation plane.

Curvatures

The surface deformations are measured on top of the reference sphere, with origin at the center of the square cell. The first two terms are the relative tilts or slopes about the *y* and the *x* axes, the third term is a relative spherical deformation, approximated by this parabolic term, since the deformation is small, the fourth term is a cylindrical (astigmatic) relative deformation with axis along the *x* or *y* axis and the last term is a cylindrical (astigmatic) relative deformation with axis at ±45^0. Thus, the transverse aberrations are given by:

$$
\begin{aligned}
-\frac{2}{r_w}TA_x &= A_1 + 2A_3x + 2A_4x + A_5y \\
-\frac{2}{r_w}TA_y &= A_2 + 2A_3y - 2A_4y + A_5x
\end{aligned}
\tag{5.28}
$$

It is possible to prove that if the eight transverse aberrations, two at each corner of the square cell are measured in principle eight coefficients can be determined. However, the eight measurements are not all independent, but there is enough information to find an accurate solution for tilts, defocus and astigmatism, including its axis orientation (Gantes-Nuñez *et al*, 2017), obtaining:

$$
\begin{aligned}
A_1 &= -\frac{1}{4r_w}\left(TA_{x1} + TA_{x2} + TA_{x3} + TA_{x4}\right) \\
A_2 &= -\frac{1}{4r_w}\left(TA_{y1} + TA_{y2} + TA_{y3} + TA_{y4}\right) \\
A_3 &= -\frac{1}{8sr_w}\left(TA_{x1} + TA_{y1} - TA_{x2} + TA_{y2} - TA_{x3} - TA_{y3} + TA_{x4} - TA_{y4}\right) \\
A_4 &= -\frac{1}{8sr_w}\left(TA_{x1} - TA_{y1} - TA_{x2} - TA_{y2} - TA_{x3} + TA_{y3} + TA_{x4} + TA_{y4}\right) \\
A_5 &= -\frac{1}{8sr_w}\left(TA_{x1} + TA_{y1} + TA_{x2} - TA_{y2} - TA_{x3} - TA_{y3} - TA_{x4} + TA_{y4}\right)
\end{aligned}
\tag{5.29}
$$

where s is the length of one side of the square cell. Once the five coefficients are found, the first two, A_1 and A_2 are ignored and from the others, the astigmatic parameters for the relative curvatures can be calculated with Eq. 5.29.

If the aberration deformations are large, the transverse aberrations are also large and the difference between the reference surface slopes and the aberrated surface slopes may be so important that the denominator in Eqs. 5.25 or 5.26 may be quite different from one. Then, the following derivatives at the center of the square cell ($x = 0$; $y = 0$) are important:

$$\begin{aligned} \frac{\partial z_r}{\partial x} &= A_1 \\ \frac{\partial z_r}{\partial y} &= A_2 \\ \frac{\partial^2 z_r}{\partial x^2} &= 2(A_3 + A_4) \\ \frac{\partial^2 z_r}{\partial y^2} &= 2(A_3 - A_4) \\ \frac{\partial^2 z_r}{\partial x \partial y} &= 2A_5 \end{aligned} \tag{5.30}$$

Then, these derivatives are used in Eq. 5.25, obtaining the relative local curvatures as:

$$c_\alpha = \frac{2\left(A_3 + \left(A_4^2 + A_5^2\right)^{1/2} \cos 2(\alpha - \psi)\right)}{\left(1 + \left(A_1 \cos\alpha + A_2 \sin\alpha\right)^2\right)\left(1 + A_1^2 + A_2^2\right)^{1/2}} \tag{5.31}$$

Another possible method that can be used in lenses is to directly measure the sagitta values at many point over the aperture with a mechanical or optical profilometer (Schmit, Creath and Wyant, 2007) and then to obtain an analytical expression for the optical surface shape. Then, the absolute local curvatures may be obtained with the exact expression in Eq. 5.31.

Examples of some local curvature maps are illustrated in Fig. 21 using a representation where the three curvature parameters, *i.e.*, spherical component, astigmatism and cylindrical axis orientation are shown in a single map, as described by Hernández-Delgado, *et al* (2021) [30]. The sampling points must be uniformly distributed over the aperture, for example, forming square or hexagonal cells. In these maps they are located in concentric circles, each one with a number of sampling points equal to six times the ring number. The red color represents positive local curvatures and blue color represents negative local curvatures.

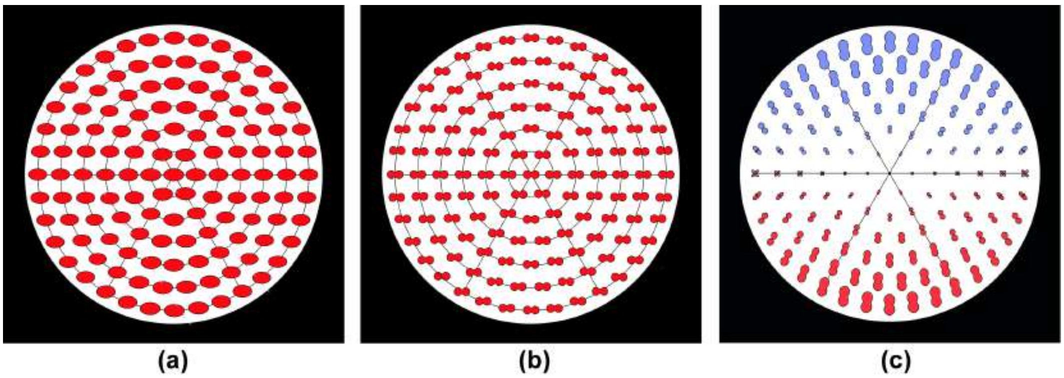


Figure 38. - Local curvature maps for an aberrated optical surface with some primary Aberrations. a) Astigmatism and defocus. b) Astigmatism and c) coma (From Hernández-Delgado, *et al* 2021).

Conclusions

The main objective of this hesis was to understand the methods to calculate the local curvature over surfaces. In the first part of the work, we analyze and construct a theory to calculate the local curvature in a three-dimensional space. The method is based on the Meusnier theorem. Using the same idea where a cosine of the angle of inclination was multiplied by the second derivate of the function, we multiply by a cosine the two-dimensional expression of the local curvature. This procedure was tested mathematically using surfaces expressed as Zernike polynomials. The main advantage of our method was the simplicity to understand the calculation and the description. Due to that, the calculation process only needs another cosine of the angle to obtain a reasonable result similar to the fundamental forms of the surfaces.

Secondly, we analyze the accuracy of our method versus the way of the Differential Geometry (first and second fundamental forms). In this part of the work, we found the expression to compensate for the minimal error in our method. The main result was that the method fits accurately in any calculation and derives only one percent error. Besides the benefit described, we use to the third part of our thesis the fundamental forms to calculate the local curvature.

Thirdly, we express and analyze the polar form of the local curvature named commonly the Euler equation. In this part, the study was focused on the polar distribution of the local curvature to different orientations. Using the principal curvatures, the Cassini ovals were graphed and used to explain the different concepts of curvatures as Gaussian, Mean, Tangential, and Sagittal curvatures.

Finally, we propose a new graphical method using the Cassini ovals to represent the local curvatures. In this part of the work, an article was published in the Optics Communications.

Future Work

As future work we plan to apply the algorithms in a wavefront analyzer, to describe any wavefront with these new curvature maps. On the other hand, the implementation of these maps in the description of free-form surfaces is seen as a field of opportunity.

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Appendix A

$$\frac{\partial}{\partial x} = \cos\theta \frac{\partial}{\partial \rho} - \frac{1}{\rho}\sin\theta \frac{\partial}{\partial \theta} \qquad ; \qquad \frac{\partial}{\partial y} = \sin\theta \frac{\partial}{\partial \rho} + \frac{1}{\rho}\cos\theta \frac{\partial}{\partial \theta}$$

$$\frac{\partial^2 f}{\partial x^2} = \left(\cos\theta \frac{\partial}{\partial \rho} - \frac{1}{\rho}\sin\theta \frac{\partial}{\partial \theta} \right)\left(\cos\theta \frac{\partial f}{\partial \rho} - \frac{1}{\rho}\sin\theta \frac{\partial f}{\partial \theta} \right)$$

$$\frac{\partial^2 f}{\partial y^2} = \left(\sin\theta \frac{\partial}{\partial \rho} + \frac{1}{\rho}\cos\theta \frac{\partial}{\partial \theta} \right)\left(\sin\theta \frac{\partial f}{\partial \rho} + \frac{1}{\rho}\cos\theta \frac{\partial f}{\partial \theta} \right)$$

$$\frac{\partial^2 f}{\partial x \partial x} = \left(\cos\theta \frac{\partial}{\partial \rho} - \frac{1}{\rho}\sin\theta \frac{\partial}{\partial \theta} \right)\left(\sin\theta \frac{\partial f}{\partial \rho} + \frac{1}{\rho}\cos\theta \frac{\partial f}{\partial \theta} \right)$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial \rho^2}\cos^2\theta + \frac{1}{\rho}\left(\frac{\partial f}{\partial \rho} + \frac{1}{\rho}\frac{\partial^2 f}{\partial \theta^2} \right)\sin^2\theta + \frac{2}{\rho}\left(\frac{1}{\rho}\frac{\partial f}{\partial \theta} - \frac{\partial^2 f}{\partial \rho \partial \theta} \right)\cos\theta \sin\theta$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial \rho^2}\sin^2\theta + \frac{1}{\rho}\left(\frac{\partial f}{\partial \rho} + \frac{1}{\rho}\frac{\partial^2 f}{\partial \theta^2} \right)\cos^2\theta - \frac{2}{\rho}\left(\frac{1}{\rho}\frac{\partial f}{\partial \theta} - \frac{\partial^2 f}{\partial \rho \partial \theta} \right)\cos\theta \sin\theta$$

$$\frac{\partial^2 f}{\partial x \partial x} = \frac{1}{\rho}\left(\frac{\partial^2 f}{\partial \rho \partial \theta} - \frac{1}{\rho}\frac{\partial f}{\partial \theta} \right)\cos 2\theta + \frac{1}{2}\left(\frac{\partial^2 f}{\partial \rho^2} - \frac{1}{\rho}\frac{\partial f}{\partial \rho} - \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \theta^2} \right)\sin 2\theta$$

Curvature

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial \rho^2}\cos^2\theta + \frac{1}{\rho}\left(\frac{\partial f}{\partial \rho} + \frac{1}{\rho}\frac{\partial^2 f}{\partial \theta^2}\right)\sin^2\theta + \frac{1}{\rho}\left(\frac{1}{\rho}\frac{\partial f}{\partial \theta} - \frac{\partial^2 f}{\partial \rho \partial \theta}\right)\sin 2\theta$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial \rho^2}\sin^2\theta + \frac{1}{\rho}\left(\frac{\partial f}{\partial \rho} + \frac{1}{\rho}\frac{\partial^2 f}{\partial \theta^2}\right)\cos^2\theta - \frac{1}{\rho}\left(\frac{1}{\rho}\frac{\partial f}{\partial \theta} - \frac{\partial^2 f}{\partial \rho \partial \theta}\right)\sin 2\theta$$

$$\frac{\partial^2 f}{\partial x \partial x} = \frac{1}{\rho}\left(\frac{\partial^2 f}{\partial \rho \partial \theta} - \frac{1}{\rho}\frac{\partial f}{\partial \theta}\right)\cos 2\theta + \frac{1}{2}\left(\frac{\partial^2 f}{\partial \rho^2} - \frac{1}{\rho}\frac{\partial f}{\partial \rho} - \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \theta^2}\right)\sin 2\theta$$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \left(\frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial f}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \theta^2}\right)$$

$$\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2} = \left(\frac{\partial^2 f}{\partial \rho^2} - \frac{1}{\rho}\frac{\partial f}{\partial \rho} - \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \theta^2}\right)\cos 2\theta - \frac{2}{\rho}\left(\frac{\partial^2 f}{\partial \rho \partial \theta} - \frac{1}{\rho}\frac{\partial f}{\partial \theta}\right)\sin 2\theta$$

$$\frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) = \frac{1}{2}\left(\frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial f}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \theta^2}\right)$$

$$\frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2}\right) = \frac{1}{2}\left(\frac{\partial^2 f}{\partial \rho^2} - \frac{1}{\rho}\frac{\partial f}{\partial \rho} - \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \theta^2}\right)\cos 2\theta - \frac{1}{\rho}\left(\frac{\partial^2 f}{\partial \rho \partial \theta} - \frac{1}{\rho}\frac{\partial f}{\partial \theta}\right)\sin 2\theta$$

$$\frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) + \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2}\right)\cos 2\alpha + \frac{\partial^2 f}{\partial x \partial x}\sin 2\alpha =$$
$$\frac{1}{2}\left(\frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial f}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \theta^2}\right) + \frac{1}{2}\left(\frac{\partial^2 f}{\partial \rho^2} - \frac{1}{\rho}\frac{\partial f}{\partial \rho} - \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \theta^2}\right)\cos 2\theta \cos 2\alpha - \frac{1}{\rho}\left(\frac{\partial^2 f}{\partial \rho \partial \theta} - \frac{1}{\rho}\frac{\partial f}{\partial \theta}\right)\sin 2\theta \cos 2\alpha +$$
$$\frac{1}{\rho}\left(\frac{\partial^2 f}{\partial \rho \partial \theta} - \frac{1}{\rho}\frac{\partial f}{\partial \theta}\right)\cos 2\theta \sin 2\alpha + \frac{1}{2}\left(\frac{\partial^2 f}{\partial \rho^2} - \frac{1}{\rho}\frac{\partial f}{\partial \rho} - \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \theta^2}\right)\sin 2\theta \sin 2\alpha$$

Curvatures

$$\frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}\right)+\frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}-\frac{\partial^2 f}{\partial y^2}\right)\cos 2\alpha+\frac{\partial^2 f}{\partial x\partial x}\sin 2\alpha=$$
$$\frac{1}{2}\left(\frac{\partial^2 f}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial f}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2 f}{\partial\theta^2}\right)+\frac{1}{2}\left(\frac{\partial^2 f}{\partial\rho^2}-\frac{1}{\rho}\frac{\partial f}{\partial\rho}-\frac{1}{\rho^2}\frac{\partial^2 f}{\partial\theta^2}\right)\cos 2(\theta-\alpha)$$
$$-\frac{1}{\rho}\left(\frac{\partial^2 f}{\partial\rho\partial\theta}-\frac{1}{\rho}\frac{\partial f}{\partial\theta}\right)\sin 2(\theta-\alpha)$$

In general the numerator:

$$\frac{1}{2}\left(\frac{\partial^2 f}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial f}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2 f}{\partial\theta^2}\right)+\frac{1}{2}\left(\frac{\partial^2 f}{\partial\rho^2}-\frac{1}{\rho}\frac{\partial f}{\partial\rho}-\frac{1}{\rho^2}\frac{\partial^2 f}{\partial\theta^2}\right)\cos 2(\theta-\alpha)-\frac{1}{\rho}\left(\frac{\partial^2 f}{\partial\rho\partial\theta}-\frac{1}{\rho}\frac{\partial f}{\partial\theta}\right)\sin 2(\theta-\alpha)$$

And the denominator:

a) Part by part, firs the sum:

$$\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2=\left(\cos\theta\frac{\partial f}{\partial\rho}-\frac{1}{\rho}\sin\theta\frac{\partial f}{\partial\theta}\right)^2+\left(\sin\theta\frac{\partial f}{\partial\rho}+\frac{1}{\rho}\cos\theta\frac{\partial f}{\partial\theta}\right)^2$$

$$\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2=$$
$$\cos^2\theta\left(\frac{\partial f}{\partial\rho}\right)^2+\frac{1}{\rho^2}\sin^2\theta\left(\frac{\partial f}{\partial\theta}\right)^2-\frac{2}{\rho}\cos\theta\sin\theta\frac{\partial f}{\partial\rho}\frac{\partial f}{\partial\theta}+\sin^2\theta\left(\frac{\partial f}{\partial\rho}\right)^2+\frac{1}{\rho^2}\cos^2\theta\left(\frac{\partial f}{\partial\theta}\right)^2+\frac{2}{\rho}\cos\theta\sin\theta\frac{\partial f}{\partial\rho}\frac{\partial f}{\partial\theta}$$

Curvature

$$\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2=\left(\frac{\partial f}{\partial \rho}\right)^2+\frac{1}{\rho^2}\left(\frac{\partial f}{\partial \theta}\right)^2$$

b) The second term:

$$\left(\frac{\partial f}{\partial x}\cos\alpha+\frac{\partial f}{\partial y}\sin\alpha\right)=\frac{\partial f}{\partial \rho}\cos\theta\cos\alpha+\frac{\partial f}{\partial \rho}\sin\theta\sin\alpha-\frac{1}{\rho}\frac{\partial f}{\partial \theta}\sin\theta\cos\alpha+\frac{1}{\rho}\frac{\partial f}{\partial \theta}\cos\theta\sin\alpha$$

$$\left(\frac{\partial f}{\partial x}\cos\alpha+\frac{\partial f}{\partial y}\sin\alpha\right)=\frac{\partial f}{\partial \rho}\left(\cos\theta\cos\alpha+\sin\theta\sin\alpha\right)+\frac{1}{\rho}\frac{\partial f}{\partial \theta}\left(\cos\theta\sin\alpha-\sin\theta\cos\alpha\right)$$

$$\left(\frac{\partial f}{\partial x}\cos\alpha+\frac{\partial f}{\partial y}\sin\alpha\right)=\frac{\partial f}{\partial \rho}\cos\left(\theta-\alpha\right)-\frac{1}{\rho}\frac{\partial f}{\partial \theta}\sin\left(\theta-\alpha\right)$$

Next:

$$\left(1+\left(\frac{\partial f}{\partial x}\cos\alpha+\frac{\partial f}{\partial y}\sin\alpha\right)^2\right)\left(1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2\right)^{1/2}$$
$$=\left(1+\left(\frac{\partial f}{\partial \rho}\cos\left(\theta-\alpha\right)-\frac{1}{\rho}\frac{\partial f}{\partial \theta}\sin\left(\theta-\alpha\right)\right)^2\right)\left(1+\left(\frac{\partial f}{\partial \rho}\right)^2+\frac{1}{\rho^2}\left(\frac{\partial f}{\partial \theta}\right)^2\right)^{1/2}$$

From that equation we can find the curvatures at different angle:

Curvatures

$$c_\alpha = \frac{\frac{1}{2}\left(\frac{\partial^2 f}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial f}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2 f}{\partial\theta^2}\right)+\frac{1}{2}\left(\frac{\partial^2 f}{\partial\rho^2}-\frac{1}{\rho}\frac{\partial f}{\partial\rho}-\frac{1}{\rho^2}\frac{\partial^2 f}{\partial\theta^2}\right)\cos 2(\theta-\alpha)-\frac{1}{\rho}\left(\frac{\partial^2 f}{\partial\rho\partial\theta}-\frac{1}{\rho}\frac{\partial f}{\partial\theta}\right)\sin 2(\theta-\alpha)}{\left(1+\left(\frac{\partial f}{\partial\rho}\cos(\theta-\alpha)-\frac{1}{\rho}\frac{\partial f}{\partial\theta}\sin(\theta-\alpha)\right)^2\right)\left(1+\left(\frac{\partial f}{\partial\rho}\right)^2+\frac{1}{\rho^2}\left(\frac{\partial f}{\partial\theta}\right)^2\right)^{1/2}}$$

Making $\alpha = \theta$:

$$c_T = \frac{\frac{\partial^2 f}{\partial\rho^2}}{\left(1+\left(\frac{\partial f}{\partial\rho}\right)^2\right)\left(1+\left(\frac{\partial f}{\partial\rho}\right)^2+\frac{1}{\rho^2}\left(\frac{\partial f}{\partial\theta}\right)^2\right)^{1/2}}$$

And $\alpha = \theta+90^0$:

$$c_S = \frac{\left(\frac{1}{\rho}\frac{\partial f}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2 f}{\partial\theta^2}\right)}{\left(1+\frac{1}{\rho^2}\left(\frac{\partial f}{\partial\theta}\right)^2\right)\left(1+\left(\frac{\partial f}{\partial\rho}\right)^2+\frac{1}{\rho^2}\left(\frac{\partial f}{\partial\theta}\right)^2\right)^{1/2}}$$

Appedix B.

Curvature

$$\Delta f(x, y) = A_1 r\cos(\theta) + A_2 r sin(\theta)$$
$$+ r^2 A_3\left(\cos(\theta)^2 + sin(\theta)^2\right) - r^2 A_4(\cos(2\theta)) - \frac{r^2 A_5}{2}(sin(2\theta))$$

$$\Delta f(x, y) = A_1 r\cos(\theta) + A_2 r sin(\theta)$$
$$+ r^2 A_3 + r^2 A_4(\cos(2\theta)) - \frac{r^2 A_5}{2}(sin(2\theta))$$

$$if\ A_5 = 0$$
$$\Delta f(x, y) = A_1 r\cos(\theta) + A_2 r sin(\theta) + r^2 A_3 + r^2 A_4(\cos(2\theta))$$
$$\Delta f(x, y) = A_0 + A_1 x + A_2 y + A_3\left(x^2 + y^2\right) + A_4\left(x^2 - y^2\right) - A_5(xy)$$

$$x = r\cos(\theta);\ y = r sin(\theta)$$

$$\Delta f(x, y) = A_1 r\cos(\theta) + A_2 r sin(\theta) + A_3\left(r\cos(\theta)^2 + r sin(\theta)^2\right)$$
$$+ A_4\left(r\cos(\theta)^2 - r sin(\theta)^2\right) - r^2 A_5(\cos(\theta) sin(\theta))$$

$$\Delta f(x, y) = A_1 r\cos(\theta) + A_2 r sin(\theta)$$
$$+ r^2 A_3\left(\cos(\theta)^2 + sin(\theta)^2\right) + r^2 A_4\left(\cos(\theta)^2 - sin(\theta)^2\right) - r^2 A_5(\cos(\theta) sin(\theta))$$
$$\Delta f(x, y) = A_1 r\cos(\theta) + A_2 r sin(\theta) + r^2 A_3 + r^2 A_4(\cos(2\theta))$$

$$\Delta f(x, y) = A_1 r\cos(\theta) + A_2 r sin(\theta) + r^2\left(A_3 + A_4(\cos(2\theta))\right)$$

$$c_\alpha = c_{sph} + c_{cyl}\cos(2\theta)$$

$$c_{sph} = A_3;\ c_{cyl} = A_4$$