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# "THEORETICAL STUDY OF QUANTUM NON-LOCAL CORRELATIONS IN INCE GAUSS BEAMS"

**Tesis que para obtener el grado de Maestro en Ciencias (Óptica)**

***Presenta: Maria Alejandra Gonzalez Dominguez***

*Director de Tesis: Dra. Laura Elena Casandra Rosales Zárate*

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# Dedication

To my grandfather, whose love will be eternal in me forever.

# Acknowledgements

First and foremost, I want to thank physics for giving me and taking everything from me. For his unique lesson in humility. Thanking always seems little, but today I want to do it: To my dad, for giving me the best gift of all: curiosity. To my mom, for always loving me. To my sister Paola, for her constant support. To my younger sisters, Valeria, Laura and María José, for inspiring me to be better for them. To my grandparents for loving me wherever, whenever and everything necessary. To all my Gonzalez family for giving me strength to continue. To Dr. Laura Rosales for becoming my role model as a teacher, researcher and woman. To Dr. Carmelo Rosales for not skimping on sharing his wisdom with me.

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# Abstract

In recent years, structured beams have had a great development, facilitating progress in their applications, including areas like quantum mechanics, more specifically in quantum information. For example, the applications of structured light in quantum mechanics contribute significantly in high-dimensional entanglement since these beams have more degrees of freedom due to the orbital angular momentum. In this thesis, Ince Gauss beams, the solution to the paraxial wave equation in elliptic coordinates with periodic boundary conditions, are studied. These structured beams are relevant and they are characterized by an important parameter, the ellipticity $\epsilon$, this parameter is of interest because it replicates the limiting cases of other families of structured beams. When $\epsilon \to 0$ the IGBs become the Laguerre Gauss Beams (LGB), while for $\epsilon \to \infty$ the IGBs become the Hermite Gauss Beams (HGB). On consideration of these characteristics of the IGBs, and taking into account that it is a continuous basis of the SU(2), according to the similarities with the representation of the Poincaré sphere between families of structured beams, a representation of the Wigner function for the IGBs, that will depend on the quadrature operators already known from group theory and on the coefficients between IGBs and LGBs, is obtained. When an expression for the Wigner function was gotten, a representation for a Bell inequality, Clauser Horne Shimony Holt (CHSH) is proposed. These two representations were validated in the limit $\epsilon \to 0$ reproducing the results for LGBs. The main result of this thesis, is the value for the CHSH Bell-type inequality, for different order of the IGBs, and for different values of the ellipticity. In all the cases, the Bell inequality is violated, this shows that for the families of

Ince Gauss under investigation, non-locality behaviour is presented. This is a very important characteristic, related to entanglements, which may give some clues for future research.

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# 1 Introduction

Structured light is a topic that has gained relevance in recent years and it corresponds to the process of projecting a particular pattern, modifying (structuring) properties of light such as amplitude, phase or polarization [1–3].

Some of the best known structured beams are the Hermite Gauss (HG) beams that are the Paraxial wave equation (PWE) solution in Cartesian coordinates and the Laguerre Gauss (LG) solution of PWE in polar coordinates [4].

As (LGB) and (HGB), Ince Gauss beams (IGB) are the exact solution to PWE. They were discovered in 2004 by bringing forward a natural solution by making a separation of variables based on elliptical coordinates [5].

When this separation of variables is made, a parameter called ellipticity ($\epsilon$) appears. It is found that for some limiting cases of this parameter, the Laguerre-Gauss beams ($\epsilon \rightarrow 0$) and the Hermite-Gauss beams ($\epsilon \rightarrow \infty$) can be recovered [6].

This is an interesting and powerful property of IGBs because it represents a continuous transition, that is, an exact and orthogonal family (IGB) that are infinite continuous bases between (LGB-HGB). Understanding this transition and studying the ways of expressing some families in terms of others; opens new paths to investigate new applications in already discovered LGB and HGB fields.

IGBs can be very powerful modes for study orbital angular momentum (OAM) [7]. Due to a linear combination that will be presented later, the Ince Gauss Helical modes can be achieved, as

with Laguerre Gauss helical modes. This provides conditions on the wavefront that allow for a well-defined OAM. Thanks to the parameters of the families IGB[8], LGB[9], HGB [10] they allow high dimensional quantum entanglement [11].
Next an approach to the IG modes is presented, starting with a short review of the Ince polynomials, the IGB formalism and ending with applications in which this family of structured light is currently being worked on.

In this research an approach to the IG modes is presented, firstly a short review of the Ince polynomials is given, follow by the IGB formalism. Finally applications in which this family of structured light is currently being worked on is shown.

## 1.1 Ince polynomials

Ince polynomials were presented in 1926 [12] where they are introduced, as the solution of the following equation:

$$\frac{d^2E}{d\eta^2} + \epsilon \sinh 2\eta \frac{dE}{d\eta} + (a - p\epsilon \cosh 2\eta)E = 0. \tag{1.1}$$

Equation (1.1) is a periodic linear second-order differential equation that has two families of independent solutions [5], namely, the even $C_p^m(\eta, \epsilon)$ and odd $S_p^m(\eta, \epsilon)$ Ince polynomials of order $p$ and degree $m$.
These solutions are periodic and can be expanded in finite trigonometric series. The corresponding expansions fall into four classes, according to their symmetry or antisymmetry, around $\eta = 0$ and $\eta = \pi/2$, namely,

$$C_{2n}^{2k}(\eta, \epsilon) = \sum_{r=0}^{n} A_r \cos 2r\eta \quad k = 0, ..., n \tag{2.a}$$

$$C_{2n+1}^{2k+1}(\eta,\epsilon) = \sum_{r=0}^{n} A_r \cos(2r+1)\eta \quad k = 0, \ldots, n \tag{2.b}$$

$$S_{2n}^{2k}(\eta,\epsilon) = \sum_{r=1}^{n} B_r \sin(2r)\eta \quad k = 1, \ldots, n \tag{2.c}$$

$$S_{2n+1}^{2k+1}(\eta,\epsilon) = \sum_{r=0}^{n} B_r \sin(2r+1)\eta \quad k = 0, \ldots, n \tag{2.d}$$

With these equations recurrence relations are generated that give way to the solution of the eigenvalue problem that is proposed in Eq. (1.1).

These recurrence relations are shown explicitly in [13] and a tridiagonal matrix is generated, whose eigenvalues and eigenvectors are the coefficients $A_r$ y $B_r$ and the values $a_m^p$ y $b_m^p$ respectively.

The information presented in [5] gives a very strong conclusion about these polynomials: Ince polynomials are denoted as $C_p^m(\eta,\epsilon)$ and $C_p^m(\eta,\epsilon)$, where $0 < m < p$ for even functions; $1 < m < p$ for odd functions. Indices $(p,m)$ have the same parity, i.e., $(-1)^{p,m} = 1$; and $\epsilon$ is the ellipticity parameter.

For more information about these polynomials and some mathematical properties that may be useful, the reader is refereed to reference [13].

## 1.2 Ince Gauss modes

Ince-Gaussian (IG) beams are the orthogonal exact solution of the Paraxial Wave Equation (PWE) in elliptical coordinates. A property of interest is the existence of a new parameter called ellipticity, that comes from the separation of variables when solving for PWE, this parameter is continuous. IG modes constitute the exact and continuous transition modes between Laguerre-Gaussian modes and Hermite-Gaussian modes. IG helical beams are a lineal combination of odd and even IGB. They have a well defined helical phase and they have been experimentally generated. Since the helical phase is well defined the angular momentum is also as well and this might be a powerful property for potential applications.

## 1.3 Applications

As could be seen previously, the field of IGBs has not been discovered for a long time, for this reason the applications in this subject, although very promising, are scarce. Some applications such as in resonators [14], optical vapor machines [15], atmospheric turbulence[16, 17] will be described below. One of the most striking and well-studied applications for IGBs is in the quantum mechanical formalism such as entanglement [8, 18, 19].

In [8] the OAM (Orbital angular moment) in quantum formalism for structured light is studied, specifically for an elliptical symmetry. This is important since it allows one to represent different infinite basis by just changing the value for $\epsilon$ (elipticity). This was possible to observe through simulations. Another very significant result is the study of OAM behavior and the demonstration of continuity for Ince-Gauss beams.

In [18] the entanglement with different singularities in structured light is shown. This is verified with a previously known correlation function in 2D PWE (Ince-Gauss modes). Also a method for generating a 3D correlation function was demonstrated, both correlation functions explicitly depends on the ellipticity parameter.

In [19] a method is presented where the entanglement with Spontaneous parametric down-conversion (SPDC) is enhanced for Ince-Gauss beams, using the Schmidt number (K) as entanglement witness. For different values of ellipticity (particular parameter of the IGB) the Schmidt number was investigated. The values to maximize the entanglement by values of K as observable were also studied as a function of the ellipticity. Also the entanglement as a function of the spiral bandwidth and OAM was investigated.

The formalism developed for Partially coherent Ince – Gaussian beams is developed [20, 21] This is of particular interest because the fluctuations that occur may be applicable to topics such as atmospheric turbulence.

Specifically, investigating atmospheric turbulence might affect optical communications. Structured light can be used for optical communications, however it is important to investigate the effects of turbulance in this beams [16]. In [17] the case of IGB OAM entanglement under atmospheric turbulence conditions is studied, this gives a much more realistic picture for experimentation and encoding.

On the other hand, interactions of light and atoms are very important. In this line of research [15] the optical storage of IGB modes are studied in a warm Rubidium vapor cell based on electromagnetically induced transparency protocol. The response of this method is qualitatively studied for the effects due to atomic diffusion.

## 1.4 Outlook

The field of IGBs is currently emerging and it is relatively new, which is why many applications are proposed and additionally, many theoretical developments in quantum formalism are being investigated due to how powerful this orthogonal family is. One of the topic of research is the non-local and entanglement properties of these beams. Future research is expected to greatly focus on IGB, taking advantage of the fact that it is a general case of the particular beams that have been studied for a longer time.

This thesis sumarizes the research carried out for the nonlocality properties of IGM, in particular the violation of the CHSH inequality. This document is organized as follows: In the present chapter IGBs have been introduced as well as their potential applications. In chapter [2] preliminary concepts of structured light, which are important to the research, are presented. The description of the Wigner function is given in [3], while in [4] Bell inequalities and non-local concepts are shown. The main results of this thesis are given in chapter [5], which include the representation of the Wigner function, as well as the proposed Bell inequality for the IGBs. Conclusions are presented in Chapter [6]. Finally, two appendices are included. In the first one a different visualization of Wigner Function of the LGBs is presented, taking account a transformation in cylindrical coordinates, interesting behavior can be identified. In the second one, a representation of momentum operator for elliptical coordinates is investigated and proposed.

# 2 Structured Light

Structured beams receive their name from the property of being able to 'structure' or modify their amplitude or phase. Actually, all light beams are structured and technology has allowed researchers to change the shape of light and thus create new applications, for example in the world of optical telecommunications.

## 2.1 Gaussian Beams

The most natural form of structured light is the Gaussian beams. Experimentally it is possible to see this in laser's setups, that is why the most realistic 'light beam' in a laboratory is a Gaussian profile. Its behaviour obeys the following properties: its width expands upon propagation, while maintaining its intensity pattern, the Gouy phase endows it with a longitudinal phase dependence, and wavefront curvature leads to a transverse radial phase dependence. The whole family of 'beams' can be described with the wave equation, in this case the paraxial wave equation (PWE) is [2]:

$$\left(\nabla_t^2 + 2ik\frac{\partial}{\partial z}\right)\Psi(\vec{r}) = 0, \tag{2.1}$$

where $\nabla_t^2$ is the transverse Laplacian, $\vec{r}$ is the position vector, and $k$ is the wave number. The lowest order of PWE solution of Gaussian beams in general is:

$$\Psi_G(\vec{r}) = \frac{w_0}{w(z)}\exp\left[\frac{-r^2}{w^2(z)} + i\frac{kr^2}{2R(z)} - \psi_{GS}(z)\right]. \tag{2.2}$$

Here $r$ is the radius, $w(z)^2 = w_0^2(1 + z^2/z_R^2)$ is the beam width, $R(z) = z + z_R^2/z$ is the radius of curvature of the phase front, $\psi_{GS}(z) = \arctan(z/z_R)$ is the Gouy shift, $z_R = kw_0^2/2$ is the Rayleigh range and $w_0$ is the beam width at $z = 0$.

This PWE can be solved in different coordinate systems, among these recognized families of orthogonal solutions are Hermite Gauss Beams (HGB), Laguerre Gauss beams (LGB) and Ince Gauss beams (IGB), which represent a Gaussian profile as in Eq. (2.2) times a polynomial's family.

### 2.1.1 Hermite Gauss beams (HGB)

The simplest solution for PWE is in Cartesian coordinates and after a variable separation's process of Eq. (2.1), HGB are represented as Gaussian profile times Hermite polynomials of order $m$ in $x$ and order $n$ in $y$, with $a$ slightly modified Gouy phase.

The Gouy phase plays a fundamental role in understanding the propagation of Gaussian waves in general, it represents a delay in the phase when the wave propagates and has a maximum value in $\pi$ [22].

The Hermite Gauss Beams take the following form:

$$\begin{aligned}\Psi_{m,n}^{HG} = \frac{2^{-N/2}}{w(z)}\sqrt{\frac{2}{\pi n!m!}}\exp\left[\frac{-ik(x^2+y^2)}{2R(z)}\right]\exp\left[\frac{-(x^2+y^2)}{w^2}\right] \\ \exp\left[-i(n+m+1)\psi\right] H_m\left(\frac{x\sqrt{2}}{w}\right) H_n\left(\frac{y\sqrt{2}}{w}\right).\end{aligned} \tag{2.3}$$

HGBs are widely known and have many powerful applications (will be discussed later) that have been worked on for a long time, among their peculiarities is the fact that they are 'countable' just like LGB and IGB.

Due to the mathematical property of linear superposition, which will be explained in detailed in Section 2.1.3, the Hermite, Laguerre and Ince polynomials can be expressed in terms of each other, as an expansion. This represents an advantage at the moment of studying other beams, since the HGB, being the solution to the PWE in Cartesian coordinates, is separable is in $x$ and $y$, and this is powerful for some applications.

### 2.1.2 Laguerre Gauss beams (LGB)

The Laguerre Gauss beams are the solution in cylindrical coordinates of PWE $(\rho, \phi, z)$, they correspond to the transformation of coordinates such that: $x = \rho \cos\phi$, $y = \rho \sin\phi$, $z$. LGBs can be expressed as a Gaussian profile times an associated Laguerre polynomial $l, p$. The Gouy phase will depend on $l$ and $p$ as well.

The Laguerre Gauss Beams take the following form:

$$\Psi_{lp}^{LG} = \frac{w_0}{w(z)} \left[ \frac{\sqrt{2}\rho}{w(z)} \right]^l L_p^l \left( \frac{2\rho^2}{w^2(z)} \right) \exp\left[ -\frac{\rho^2}{w^2(z)} \right] \exp\left[ \frac{ik\rho}{2R(z)} \right] \exp[-i(2p+l+1)\xi(z)] \exp[il\phi]. \tag{2.4}$$

LGBs are very powerful because their changes in the phase represent a helical wavefront, this means that the phase varies azimuthally, this means in turn that there is a well-defined Orbital Angular Moment (OAM) which opens the doors to many applications but also in the theoretical field for quantum formalism.

### 2.1.3 Ince Gauss Beams

IGB were presented in 2004 [5] as the natural solution of the PWE. A solution is proposed so for a $z$-propagating wave: $U = \Psi(\xi, \eta, z)\exp(ikz)$ where $(\xi, \eta)$ are the transverse coordinates, while the PWE is given in Eq. (2.1)

Next, a proposed solution for elliptical coordinates is given by:

$$IG(\vec{r}) = E(\xi)N(\eta)\exp[iZ(z)]\Psi_G(\vec{r}), \tag{2.5}$$

where $E$, $N$ and $Z$ are real functions, and in the transverse $z$-plane the elliptic coordinates are defined as: $x = f(z) \cosh \xi \cos \eta$, $y = f(z) \sinh \xi \sin \eta$ and $z = z$ where $\xi \in [0, \inf)$ and $\eta \in [0, 2\pi)$ are the radial and the angular elliptic variables, respectively. $f(z) = f_0 w(z)/w_0$ due to the fact that semifocal separation $f$ diverges in the same way as the width of the GB; where $f(z)$ is the semifocal separation at the waist plane $z = 0$.

In this way, on substituting the proposed solution given in Eq. (2.5) in Eq. (2.1) the following differential equations are obtained:

$$\frac{d^2E}{d\xi^2} - \epsilon \sinh 2\xi \frac{dE}{d\xi} - (a - p\epsilon \cosh 2\xi)E = 0, \tag{2.6}$$

$$\frac{d^2E}{d\eta^2} + \epsilon \sinh 2\eta \frac{dE}{d\xi} + (a - p\epsilon \cosh 2\eta)E = 0, \tag{2.7}$$

$$-\left(\frac{z^2 + z_R^2}{z_R}\right)\frac{dZ}{dz} = p. \tag{2.8}$$

It is easily seen that Eqs. (2.6) and (2.7) are the same as Eq. (1.1), so the Ince polynomials of order $p, m$ and ellipticity $\epsilon = 2f_0^2/w_0^2$ can be used as a solution. Then on multiplying the expression of the Gaussian nature, Eq. (2.2), with the Ince polynomials the following expressions are obtained:

$$IG^e_{p,m} = \frac{Cw_0}{w(z)} C_p^m(i\xi, \epsilon) C_p^m(\eta, \epsilon) \exp[\frac{-r^2}{w^2(z)}] \exp\left[ikz + \frac{kr^2}{2R(z)} - (p+1)\psi_{GS}(z)\right], \tag{2.9}$$

$$IG^o_{p,m} = \frac{Sw_0}{w(z)} S_p^m(i\xi, \epsilon) S_p^m(\eta, \epsilon) \exp[\frac{-r^2}{w^2(z)}] \exp\left[ikz + \frac{kr^2}{2R(z)} - (p+1)\psi_{GS}(z)\right]. \tag{2.10}$$

In figures (2.1) and (2.2) the intensities of even and odd IGB are shown, for the transverse plane with $z = 0$ and $\epsilon = 2$. These figures were plotted using $0 < p < 4$ and $0 < m < 3$.

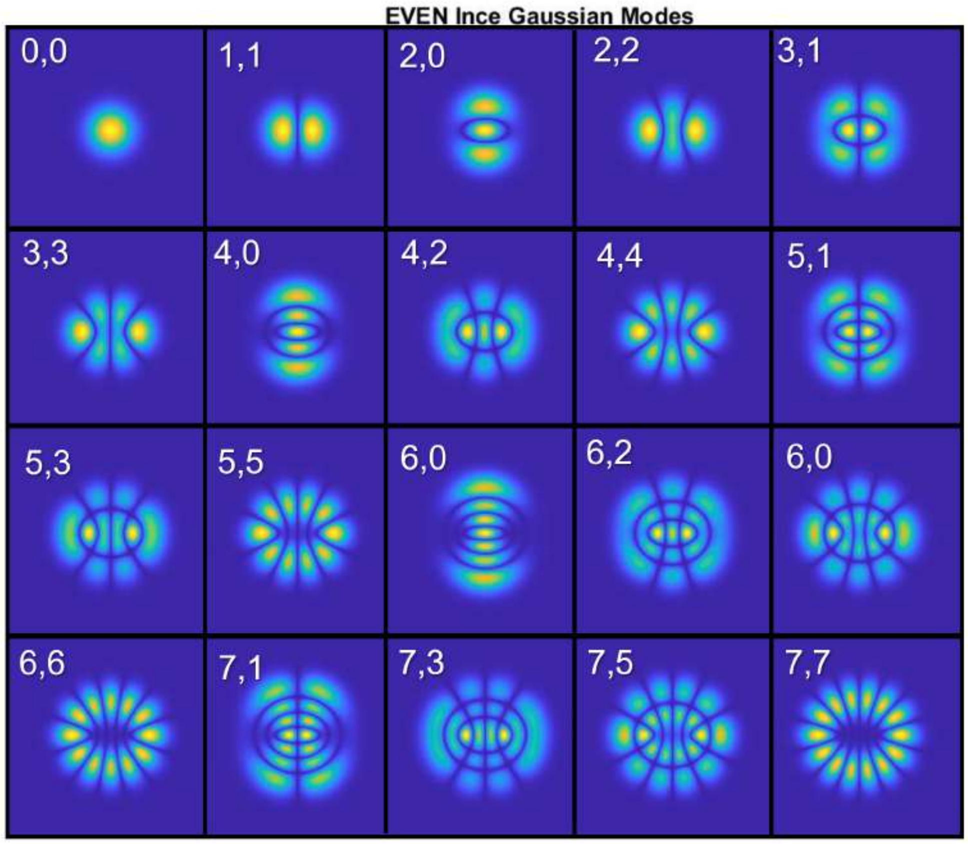


**Figure 2.1:** Intensity patterns of even Ince Gauss modes with $\epsilon = 2$, where p and are the order of the Ince polynomials

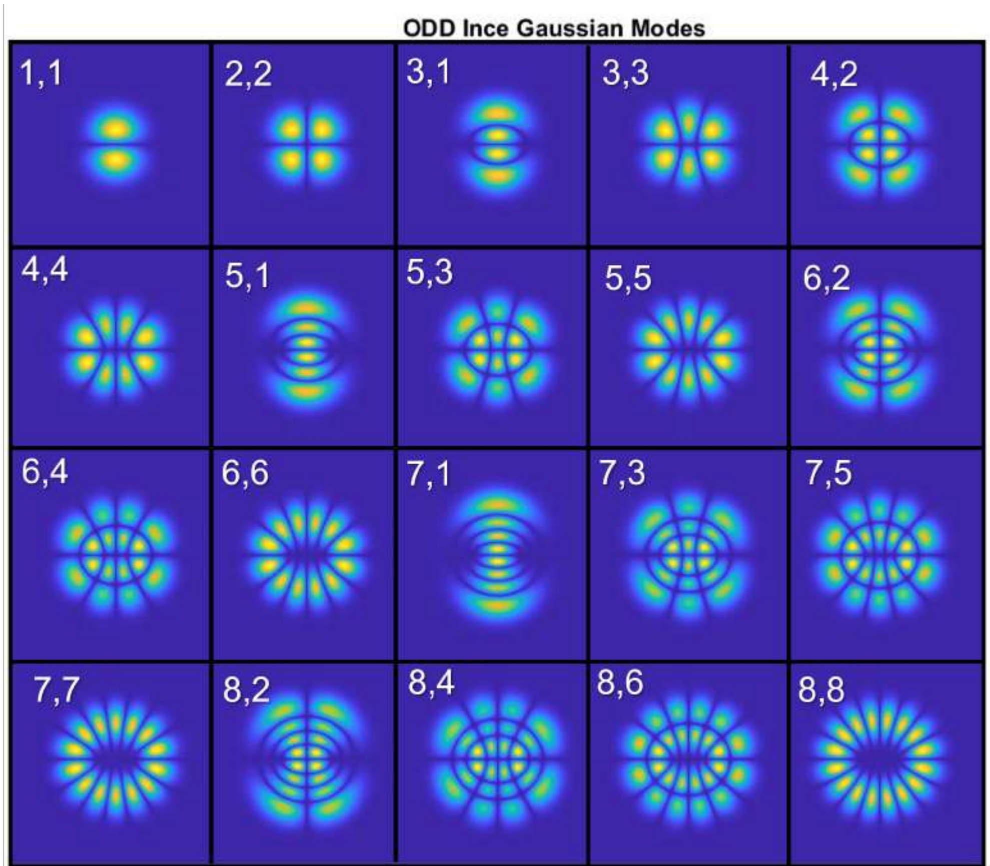


**Figure 2.2:** Intensity patterns of odd Ince Gauss modes with $\epsilon = 2$, where p and are the order of the Ince polynomials

As mentioned before, one of the most interesting properties about IGBs is that they represent the transition between HGB and LGB. This means that by varying the parameter $\epsilon$ a different orthogonal family can be obtained, and in the limiting cases the HGB families and their parameters $(n_x, n_y)$ and LGB and their parameters $(n, l)$ are recovered. In order to recreate the IGB beams $(p, m)$ one can use the following equivalences. IGB Even $\longrightarrow$ HGB: $n_x = m$ and $n_y = p - m$; IGB odd $\longrightarrow$ HGB: $n_x = m - 1$ and $p - m + 1$. IGB $\longrightarrow$ LGB: $m = l$ and $p = 2n + 1$.

In figure (2.3) a transition between odd LGB $\longrightarrow$ IGB $\longrightarrow$ HGB using $\epsilon = 2$ and $p = 5$ is shown.

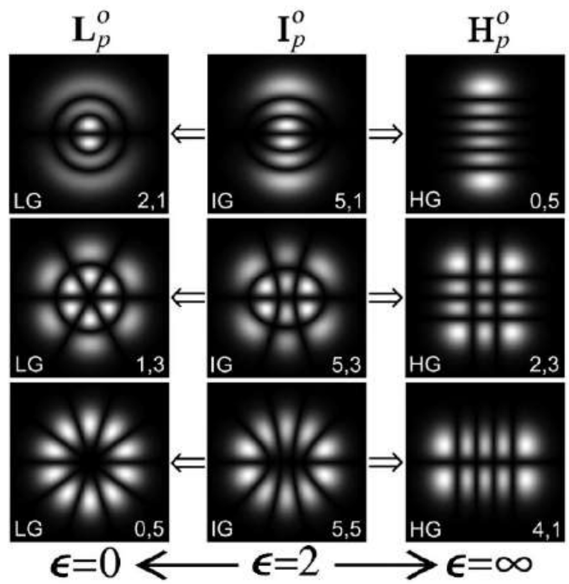


**Figure 2.3:** Transition between odd LGB → IGB → HGB with $\epsilon = 2$. Figure taken from [5].

Additionally, the orthonormality relation for the IGB is known given by:

$$\int\int_{-\infty}^{\infty} IG_{p,m}^{\sigma}\overline{IG}_{p,m}^{\sigma} = \delta_{\sigma\sigma'}\delta_{pp'}\delta_{mm'} \tag{2.11}$$

In [23] one way to express the IGB in terms of the LGB is shown. Beginning with the fact that they can be expressed as a superposition of modes as follows:

$$LG_{n,l}^{\sigma}(r,\phi) = \sum_m D_m IG_{p=2n+l,m}^{\sigma}(\xi,\eta,\epsilon) \tag{2.12}$$

$$IG_{p,m}^{\sigma}(\xi,\eta,\epsilon) = \sum_{l,n} D_{l,n}^{\sigma} LG_{n,l}(r,\phi) \tag{2.13}$$

where $\sigma = e, o$ is the parity. The coefficients $D$ correspond to the overlap integral between an IG and a LG and can be obtained by applying group theory techniques [23]:

$$\int_{-\infty}^{\infty} LG_{n,l}^{\sigma}\overline{IG}_{p,m}^{\sigma} = \delta_{\sigma'\sigma}\delta_{p,2n+l}(-1)^{n+l+(p+m)/2}\sqrt{(1+\delta_{0,l})\Gamma(n+l+1)n!}A\sigma_{(l+\delta_{0,\sigma})/2}(a_p^m) \tag{2.14}$$

The equivalence of IGB $\longrightarrow$ HGB can be deduced from the equivalence between LGB $\longrightarrow$ HGB. LGMs with azimuthal angular dependence $\exp(\pm il\phi)$ have a phase that rotates circularly about the propagation axis.

For the IG modes, a similar construction can be made that takes the name of Helical Ince Gauss Beams (HIGB) and is given by:

$$HIG^{\pm}_{p,m} = IG^{e}_{p,m}(\xi, \eta, \epsilon) \pm iIG^{o}_{p,m}(\xi, \eta, \epsilon) \tag{2.15}$$

Eq. (2.15) is valid for $m > 0$ because $IG^{o}_{p,m}$ is not defined for $m = 0$.

Other beams that are also important are the Bessel beams [24, 25], the Mathieu beams [26], and the Airy beams [2]. Those beams have special characteristics as infinite energy in their theoretical version and numerous applications [27].

## 2.2 Experimental generation of structured light

The experimental generation of these beams is done in a similar way to the other families of structured beams, which is through the projection of a hologram on a spatial light modulator (SLM) or with Digital micromirror device (DMD) technology[28]. In this way the wave front acquires ‘structure’, making possible to study these beams [29]. As scalar structured beams, HGBs, LGBs, IGBs have been studied, the generation of vector beams by means of SLM [30] and Digital micromirror device (DMD) [31] has also been studied.

## 2.3 Applications

Some of the IGB applications were discussed in the previous chapter, however, more general ones will be presented below that gives an idea about the importance of structured light for current applied optics research.

For quantum applications probably the most powerful property of structured beams is orbital angular momentum (OAM). In 1992 it was shown that light can carry OAM due to the azimuthal variation of the phase ($e^{il\phi}$) where $l$ corresponds to the units of OAM [32].

Particularly, for families of solutions such as LGB or BGB and even IGB, application with multidimensional entanglement can stand out due to the azimuthal variation of the phase [11]. The continuity of the parameter $l$ plays a fundamental role, because being discrete but in a range of $(-\infty, \infty)$ opens the possibility of investigation about high dimensional entanglement and not only bipartite entanglement as it has been previously done [33].

Although, bipartite entanglement has been presented in families of HGB solutions, the LGB, having a well-defined OAM, opened the possibilities to carry a value of $l\hbar$, as it is an orthogonal base, infinite and discrete modes can be generated that can be interlaced between them.

Upon discovering the robust properties of OAM in entanglement the next step is generating larger number's states. Up to this date, as far as is known a value of $10,010$ is reported in the literature [34], which opened the doors for significant applications in quantum communication [35].

Some of the advantages of high-dimensional entanglement are: higher information capacity since the qudits can contain additional information than qubits (bipartite systems [36], enhanced robustness against eavesdropping and quantum cloning [37], quantum communication without monitoring signal disturbance [38] and larger violation of Bell inequality [39].

Some of the most popular applications in the area of quantum optics for Gaussian beams are in: High-dimensional criptography [40], Quantum Key Distribution (QKD) in OAM [38, 41], quantum cloning [42, 43], quantum teleportation [44, 45], high dimensional quantum gates [46], Wigner functions in twisted photons [47, 48], Bell inequalities in experimental high dimensional entanglement [49]. Another specific references can be found in [11, 50, 51].

# 3 | Wigner function

The criteria for defining a limit between the study of light in a classical or non-classical way go back to the very formalism of quantum mechanics, one of these criteria corresponds to the exploitation of the properties of probability distributions in phase space.

## 3.1 Density matrix

Historically, the first function of this type was proposed in 1932 by E. Wigner and is known as the Wigner function (WF), which is not exactly a probability distribution, but a quasi-probability distribution, because for some states it can take negative values [52].

For the understanding and interpretation of the WF, we will begin with the description of the density matrix $\hat{\rho}$, which is defined as follows:

$$\hat{\rho} = \sum_n \rho_n \left|\psi_n\right\rangle \left\langle\psi_n\right| . \tag{3.1}$$

Here $|\psi_n\rangle$ represents the state vector in the Dirac notation, in this way the following conditions can be imposed for the pure and mixed states.

- Pure states: There is only one state then $\rho_n = 1$ and $Tr[\hat{\rho}^2] = 1$.
- Mixed states: $Tr[\hat{\rho}^2] \leq 1$

Note that the trace of $\hat{\rho}^2$ is the criterion to define whether a state is pure or mixed.

## 3.2 Different representations of the Wigner function

The common representation of the WF is:

$$W_{\hat{\rho}}(q,p) = \frac{1}{2\pi\hbar}\int \left\langle q - \frac{x}{2}\middle|\hat{\rho}\middle|q + \frac{x}{2}\right\rangle \exp(ipx/\hbar)dx. \tag{3.2}$$

In the case that $\hat{\rho}$ represents a pure state, the WF is:

$$W_{\psi}(x,p)\frac{1}{2\pi\hbar}\int \psi\left(q - \frac{x}{2}\right)\psi^*\left(q + \frac{x}{2}\right)\exp(ipx/\hbar)dx, \tag{3.3}$$

and the normalization is given by:

$$\int dx \int dp W_{\psi}(x,p) = 1. \tag{3.4}$$

The Wigner function is real but it can be negative, this is the reason why it cannot be considered as a genuine probability distribution. However, when it is integrated over one of the two conjugate variables (ex. $x - p$), the probability distribution of the other is obtained, that is:

$$\int dp W_{\psi}(x,p) = |\psi|^2. \tag{3.5}$$

In the formulation of probabilities and quasi-probabilities in phase space, a powerful concept is required to understand the pertinent transformations, this is the characteristic function that is given by [53]:

$$C(k) = \int dx \exp(ikx)\rho(x), \tag{3.6}$$

when $\rho(x)$ is a classical probability, in a quantum formalism the $C(k)$ can be written as:

$$C_W(\lambda) = Tr[\hat{\rho}\exp(\lambda\hat{a}^\dagger - \lambda^*\hat{a})], \tag{3.7}$$

this is a specific representation for the WF. Then the WF in terms of characteristic function can be expressed by:

$$\begin{aligned} W(\alpha) &= \frac{1}{\pi^2}\int \exp(\lambda^*\alpha - \lambda\alpha^*)C_W(\lambda)d^2\lambda \\ &= \frac{1}{\pi^2}\int \exp(\lambda^*\alpha - \lambda\alpha^* Tr[\hat{\rho}\exp(\lambda\hat{a}^\dagger - \lambda^*)\hat{a})]d^2\lambda. \end{aligned} \tag{3.8}$$

The original formulation of the Wigner function can be written as [54]:

$$\bar{W}(\bar{x},\bar{p}) = \frac{1}{(2\pi)^2}\int\int d\sigma d\mu Tr\{\rho\exp(i\mu(x-\bar{x}) + i\sigma(p-\bar{p}))\}. \tag{3.9}$$

Considering:

$$\left\langle \{p^n x^m\}_{sym} \right\rangle = \int\int d\bar{p}d\bar{x}\bar{W}(\bar{x},\bar{p})\bar{p}^n\bar{x}^m, \tag{3.10}$$

and using Eq. (3.8), with $\alpha = \frac{i\bar{p}}{\hbar}\frac{1}{\sqrt{2\eta}} + \bar{x}\sqrt{\frac{\eta}{2}}$ it is possible to formulate:

$$\bar{W}(\bar{x},\bar{p}) = \frac{1}{2\hbar}W(\alpha,\alpha^*). \tag{3.11}$$

Now it is necessary to assume the following definitions:

- $Tr\{A\} = \int dx \langle x| A |x\rangle$,
- $\exp(-ipa/\hbar)|x\rangle = |x+a\rangle$,
- $\exp(i\mu x + i\sigma p) = \exp(i\mu x)\exp(i\sigma p)\exp(-i\hbar\sigma\mu/2)$.

Finally, getting to:

$$W(\bar{x},\bar{y}) = \frac{1}{\pi\hbar}\int dy \, \langle \bar{x}+y| \, \rho \, |\bar{x}-y\rangle \exp(-2iy\bar{p}/\hbar), \tag{3.12}$$

which is the same expression that the one given in Eq. (3.2).

Another representation for the Wigner function is in terms of the displacement operator $\hat{D}(\alpha) = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})$ and is given by:

$$W(\alpha) = 2Tr[D(-\alpha)\rho D(\alpha)P], \tag{3.13}$$

where $P = \exp(i\pi\hat{a}^\dagger\hat{a})$. It can also be written as [55]:

$$W(\alpha,\beta) = \hat{D}_a(\alpha)\hat{D}_b(\beta)(-1)^{\hat{n}_a+\hat{n}_b}\hat{D}_a^\dagger(\alpha)\hat{D}_b^\dagger(\beta). \tag{3.14}$$

The WF can be written as a quasi-probability in quantum formalism. In fig. (3.1) the WF for three different states is shown, these states are the coherent state $|\alpha\rangle$, the number state $|n\rangle$ and the Schrödinger's cat state $|\alpha_+\rangle = N(|\alpha\rangle + |-\alpha\rangle)$. Note that the WF has negative values for b) $|n\rangle$ and c) $|\alpha_+\rangle$ which is an evidence about the non-classical nature of these states.

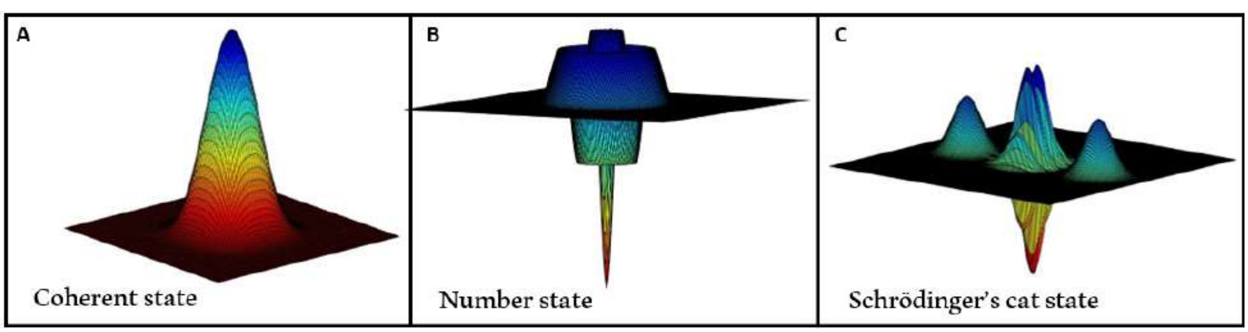


**Figure 3.1:** Wigner function for a) Coherent state; b) Number state; c) Schödinger's cat state.

Three different WFs for different states were shown in Fig (3.1). In this thesis the structured light take an important role, accordingly in the following section the WF for Gaussian beams

(LGB and HGB) will be described.

## 3.3 Wigner representation of Laguerre-Gaussian beams

In order to find a Wigner representation for the LG beams, the WF for HG beams, which is already known, will be used as starting point to describe the WF for LG. This is possible since there are interesting properties between Laguerre and Hermite polynomials [56], this is given below.

The family of solutions in one dimension for the HGB, as it was mention in Section (2.1.1), is:

$$\psi_n = \left( \frac{\sqrt{2}}{\sqrt{\pi} 2^n w n!} \right)^{1/2} H_n(\sqrt{2}x/w) \exp(-x^2/w^2). \tag{3.15}$$

The WF for the one dimensional HGB is [56]:

$$W_n(x, p_x) = \frac{(-1)^n}{\pi \lambda\!\!\!^-} Ln \left[ 4 \left( \frac{x^2}{w^2} + \frac{w^2 p_x^2}{4\lambda\!\!\!^-{}^2} \right) \right] \exp \left[ -2 \left( \frac{x^2}{w^2} + \frac{w^2 p_x^2}{4\lambda\!\!\!^-{}^2} \right) \right]. \tag{3.16}$$

Here $\lambda\!\!\!^- = \frac{\lambda}{2\pi}$ and $w$=beam waist. In this work the interest is in 2D HGB so the following separable function is proposed:

$$\psi_{n_1,n_2} \equiv \psi_{n1}(x)\psi_{n2}(y). \tag{3.17}$$

Since the WF for this case can also be expressed in a separable form, this is given by:

$$W^{HG}_{n_1,n_2}(\mathbf{x}, \mathbf{p}) = W_{n_1}(x, p_x) \times W_{n_2}(y, p_y), \tag{3.18}$$

in both cases $n_1, n_2 = 0, 1, ...$

On substituting Eqs. (3.16) and (3.17) in Eq. (3.18), and after arranging terms we get the Wigner representation for the HG beams in terms of $n_1, n_2, p_x, p_y$

$$W_{n_1,n_2}^{HG}(\mathbf{x},\mathbf{p}) = \mathcal{N} L_{n_1}\left[4\left(\frac{x^2}{w^2}+\frac{w^2p_x^2}{4\lambda^2}\right)\right] L_{n_2}\left[4\left(\frac{y^2}{w^2}+\frac{w^2p_y^2}{4\lambda^2}\right)\right]\exp\left[-2\left(\frac{x^2+y^2}{w^2}+\frac{w^2(p_x^2+p_y^2)}{4\lambda^2}\right)\right]. \tag{3.19}$$

Here $\mathcal{N} = \frac{(-1)^{n_1+n_2}}{\pi\lambda}$. Now to find the Wigner function the two-dimensional function of the LG beams should be considered, unfortunately this function is not separable as the HG beams are, that is why a different procedure to find the Wigner function will be used for this case. From Eq. (2.4) the following representation of LGB can be obtained:

$$\Phi_{j,m}^{LG}(\mathbf{x}) = \sqrt{\frac{2}{\pi w^2}}\left[\frac{(j-|m|)!}{(j+|m|)!}\right]^{1/2}\left(\frac{\sqrt{2}\mathbf{x}}{w}\right)^{2|m|}\exp(i2m\theta)L_{j-|m|}^{2|m|}\left(\frac{2\mathbf{x}^2}{w^2}\right)\exp\left(\frac{-\mathbf{x}^2}{w^2}\right). \tag{3.20}$$

Considering $\mathbf{x} = (r\cos(\theta), r\sin(\theta))$, $j = 0, 1/2, 1, \ldots$, and $m = j, j-1, \ldots, -j$.

It is possible to write the LGB as a lineal combination of the HGB as:

$$\Phi_{j,m}^{LG}(\mathbf{x}) = \sum_{n_1,n_2} C_{j,m;n_1,n_2}\Psi_{n_1,n_2}^{HG}(\mathbf{x}). \tag{3.21}$$

From quantum mechanics the following operators are proposed:

$$\hat{x} : \psi(\mathbf{x}) \to x\psi(\mathbf{x}), \hat{p_x} : \psi(\mathbf{x}) \to \frac{\lambda}{i}\frac{\partial}{\partial x}\psi(\mathbf{x}), \tag{3.22}$$

$$\hat{y} : \psi(\mathbf{x}) \to y\psi(\mathbf{x}), \hat{p_y} : \psi(\mathbf{x}) \to \frac{\lambda}{i}\frac{\partial}{\partial y}\psi(\mathbf{x}). \tag{3.23}$$

An extended vector $\hat{\xi} = (\hat{x}, \hat{y}, \hat{p_x}, \hat{p_y})$, can be used to define a unitary transformation in the Hilbert space as:

$$S : W(\xi) \to W'(\xi) = W(S^{-1}\xi).$$

Additionally, the following operators or transformations can be defined using the operators given in Eqs. (3.22), (3.23)

$$\hat{T}_0 = [\alpha^{-1}(\hat{x}^2 + \hat{y}^2) + \alpha(\hat{p}_x^2 + \hat{p}_y^2)]/4\lambdabar - 1/2,$$
$$\hat{T}_3 = [\alpha^{-1}(\hat{x}^2 - \hat{y}^2) + \alpha(\hat{p}_x^2 - \hat{p}_y^2)]/(4\lambdabar),$$
$$\hat{T}_1 = (\alpha^{-1}\hat{x}\hat{y} + \alpha\hat{p}_x\hat{p}_y)/(2\lambdabar),$$
$$\hat{T}_2 = (\hat{x}\hat{p}_y - \hat{y}\hat{p}_x)/(2\lambdabar). \tag{3.24}$$

These operators satisfy the following commutation relations $[\hat{T}_a, \hat{T}_b] = i\epsilon_{abc}\hat{T}_c,\ [\hat{T}_a, \hat{T}_0] = 0$.

After some algebraic operations the action of the operators, given in Eq. (3.24), into the HGB gives the following eigenvalue equations where the eigenvalues are given in terms of quantum numbers $n_1, n_2$:

$$\hat{T}_0\Psi^{HG}_{n_1,n_2}(\mathbf{x}) = \frac{n_1 + n_2}{2}\Psi^{HG}_{n_1,n_2}(\mathbf{x}), \tag{3.25}$$
$$\hat{T}_3\Psi^{HG}_{n_1,n_2}(\mathbf{x}) = \frac{n_1 - n_2}{2}\Psi^{HG}_{n_1,n_2}(\mathbf{x}). \tag{3.26}$$

In the case of the LGB the eigenvalue equations is given in terms of the quantum numbers $m$ and $j$ as is shown below:

$$\hat{T}_0\Phi^{LG}_{j,m}(\mathbf{x}) = j\Phi^{LG}_{j,m}(\mathbf{x}), \tag{3.27}$$
$$\hat{T}_2\Phi^{LG}_{j,m}(\mathbf{x}) = m\Phi^{LG}_{j,m}(\mathbf{x}). \tag{3.28}$$

In particular, it is found that for $\hat{T}_1$ the following relations holds

$$\hat{\xi} \longrightarrow \exp(i2\phi\hat{T}_1)\hat{\xi}\exp(-i2\phi\hat{T}_1) = S_1(\phi)\hat{\xi},$$

where $S_1(\phi)$ is defined by:

$$S_1(\phi) = \begin{bmatrix} c & 0 & 0 & \alpha s \\ 0 & c & \alpha s & 0 \\ 0 & -\alpha^{-1} s & c & 0 \\ -\alpha^{-1} s & 0 & 0 & c \end{bmatrix}. \tag{3.29}$$

In this way, the following equivalence for the operators ($\hat{T}_0$ and $\hat{T}_2$) can be proposed as:

$$\exp\left(i\frac{\pi}{2}\hat{T}_1\right)\hat{T}_0 \exp\left(-i\frac{\pi}{2}\hat{T}_1\right) = \hat{T}_0, \tag{3.30}$$

$$\exp\left(i\frac{\pi}{2}\hat{T}_1\right)\hat{T}_3 \exp\left(-i\frac{\pi}{2}\hat{T}_1\right) = \hat{T}_2. \tag{3.31}$$

Using these operators the LG polynomials can be expressed as a function of HG polynomials in the following way:

$$\Phi^{LG}_{j,m}(\mathbf{x}) = \exp\left(i\frac{\pi}{2}\hat{T}_1\right)\Psi^{HG}_{n_1,n_2}(\mathbf{x}). \tag{3.32}$$

Therefore the Wigner function for LGB can be written by using the Wigner function for HG applying a transformation of $S_1(-\pi/4)^{-1}$

$$W^{LG}_{j,m} = (\xi) = W^{HG}_{n_1,n_2}[S_1(-\pi/4)^{-1}\xi] \tag{3.33}$$

$$n_1 = j + m, n_2 = j - m$$

The quadratic expressions are defined as:

$$
\begin{aligned}
Q_0 &\equiv \frac{1}{2}\left[\frac{x^2+y^2}{w^2}+\frac{w^2}{4\lambda\!\!\!^{-}{}^2}(p_x^2+p_y^2)\right],\\
Q_1 &\equiv (xy+p_yp_x)/(2\lambda\!\!\!^{-}),\\
Q_2 &\equiv (xp_y-yp_x)/(2\lambda\!\!\!^{-}),\\
Q_3 &\equiv \frac{1}{2}\left[\frac{x^2-y^2}{w^2}+\frac{w^2}{4\lambda\!\!\!^{-}{}^2}(p_x^2-p_y^2)\right].
\end{aligned}
\tag{3.34}
$$

Finally, the Wigner function for the HG beams is:

$$
W^{LG}_{n_1,n_2}(\xi) = (-1)^{n_1+n_2}(\pi\lambda\!\!\!^{-})^{-1}L_{n_1}[4(Q_0+Q_3)]L_{n_2}[4(Q_0-Q_3)]\exp(-4Q_0) \tag{3.35}
$$

In order to show that the expression in Eq. (3.35) is correct, the relations given in (3.34) will be substituted on the expression. Also considering that:

$$
Q_0+Q_3 = \frac{x^2}{w^2}+\frac{w^2}{4\lambda\!\!\!^{-}{}^2}p_x^2. \tag{3.36}
$$

On the other hand,

$$
Q_0-Q_3 = \frac{y^2}{w^2}-\frac{w^2}{4\lambda\!\!\!^{-}{}^2}p_y^2. \tag{3.37}
$$

Next, on replacing these on Eq. (3.35), the following expression for the Wigner function is obtained:

$$
\begin{aligned}
W^{LG}_{n_1,n_2}(\xi) = \frac{(-1)^{n_1+n_2}}{\pi\lambda\!\!\!^{-}}L_{n_1}&\left[4\left(\frac{x^2}{w^2}+\frac{w^2}{4\lambda\!\!\!^{-}{}^2}p_x^2\right)\right]\exp\left[-2\left(\frac{x^2}{w^2}+\frac{w^2}{4\lambda\!\!\!^{-}{}^2}p_x^2\right)\right]\\
L_{n_2}&\left[4\left(\frac{y^2}{w^2}+\frac{w^2}{4\lambda\!\!\!^{-}{}^2}p_y^2\right)\right]\exp\left[-2\left(\frac{y^2}{w^2}+\frac{w^2}{4\lambda\!\!\!^{-}{}^2}p_y^2\right)\right].
\end{aligned}
\tag{3.38}
$$

Finally, one can notice that eq. (3.38) is the explicit form of eq. (3.18)

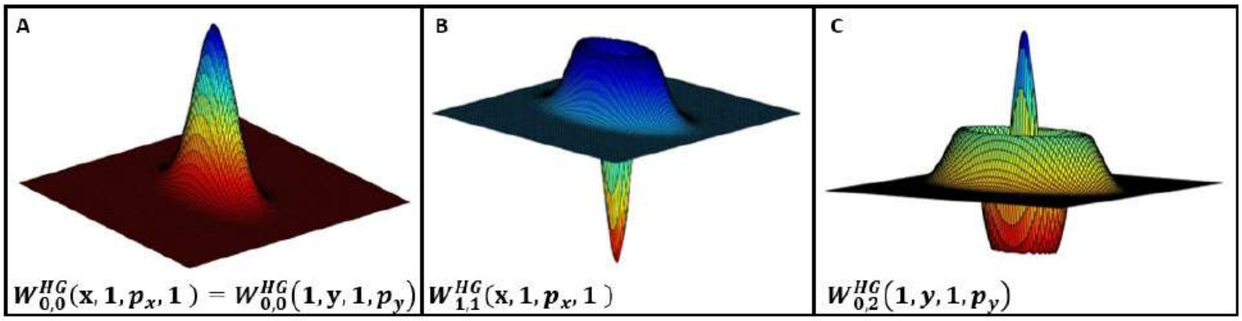


**Figure 3.2:** Wigner function for Hermite Gauss beams

As a result of replacing this into Eq. (3.33), the Wigner representation for the Laguerre Gauss beams is:

$$W^{LG}_{j,m}(\xi) = (-1)^{2j}(\pi\lambda)^{-1}L_{j+m}[4(Q_0 + Q_2)]L_{j-m}[4(Q_0 - Q_2)]\exp(-4Q_0). \tag{3.39}$$

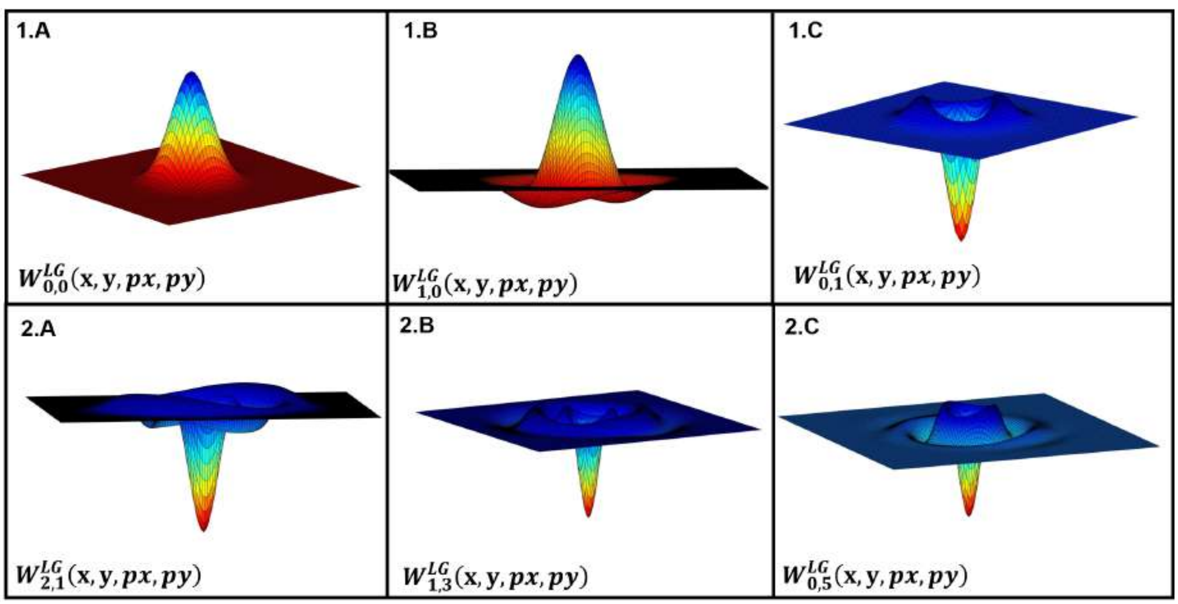


**Figure 3.3:** Wigner function for Laguerre Gauss beams

Figure 3.3 shows the Wigner function in Cartesian coordinates for different orders of the LG polynomials. These results are in agreement with the description given in [56], here the negative values indicate the non-classical nature of this light.

The above expression for the Wigner function is for an specific type of structured beam which is the LGB, however for an exact Wigner representation for optical spatial modes carrying orbital angular momentum is given in [57]. The expression proposed in [57] is found by exploiting the underlying SU(2) Lie-group algebra of their associated Poincaré sphere:

$$W_{l,p}(\zeta, \theta, \phi) = \frac{(-1)^N}{\pi^2 \lambda\!\!\!^{-2}} \exp(-Q_0) L_{N-l/2}(Q_0 - 4\mathbf{Q} \cdot \mathbf{u}_r) L_{N+l/2}(Q_0 + 4\mathbf{Q} \cdot \mathbf{u}_r). \tag{3.40}$$

Where $Q_0$ corresponds to the expression described previously in Eq. (3.34), it is worth noticing that here the term $\mathbf{Q} = (Q_1, Q_2, Q_3)$ is included, meanwhile in the LG's Wigner function it was not and $\mathbf{u}_r$ it is a unit vector on the poincaré sphere surface that describes the orbital angular momentum.

# 4 Hidden-variables theory and Bell inequality

In this chapter the interpretation about quantum mechanics as a correct and complete formalism will be discussed historically.

The boom that produced the discovery of quantum mechanics, inevitably shook the twentieth century physicists, so it was not surprising that Einstein himself, who won a Nobel Prize in physics (1921) for the discovery of the photoelectric effect, was involved in the deepest epistemological discussions of quantum mechanics.

In 1935, Einstein with Podolsky and Rosen proposed, in their article called: *Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?*, that quantum mechanics was correct but incomplete, which gave rise to something called the hidden variables theory that corresponds to explaining quantum mechanics as a completely deterministic phenomenon [58].

## 4.1 Hidden-variables theory

The hidden variables theory formulates that the phenomena described by quantum mechanics are not subject to an observer and that the Copenhagen interpretation is incorrect. Instead, it proposes a new interpretation for quantum mechanics based on two concepts:

- Locality: Here a system of two particles A and B (or two subsystems) that are separated locally (spatially) are considered, where the behavior of A does not affect that of B.
- Realism: This implies that all the elements of nature have a predetermined behavior before being measured, that is, that reality (the observables) exists before being measured.

These two premises, which are known as local realism, are the epistemological basis of hidden variables theory.

Elements of reality: These are described as all observables whose behavior is previously determined, even before being observed, that is, in the theory of hidden variables, all variables are elements of reality, including position and moment simultaneously, thus not recognizing Heisenberg's uncertainty principle.

According to all of the above, the hidden variables appear, denoted by $\lambda$, which are precisely those variables that are not easy to find or model in a physical phenomenon but which, thanks to their existence, quantum phenomena are deterministic.

*A simple example to understand this can be the toss of an uncharged coin, usually this phenomenon is studied as probabilistic having a 50/50 of the possibilities between heads and tails. But because it is a macroscopic phenomenon it is completely deterministic and not random. What happens is that to describe it completely deterministic, a modeling is required with variables such as: the gravity force, the air rosin, the rotational moment of the coin, etc. In such a way, if they could be found and studied these variables, this phenomenon could be studied as deterministic and not probabilistic, something similar suggested EPR of quantum phenomena, where these variables (the hidden variables of each problem) were needed so there is not a system governed by randomness and interpretation ambiguous of quantum mechanics.*

## 4.2 Bell inequality

Although in 1932 the renowned mathematician John Von Neumann [59] had proposed a mathematical proof against the hidden-variables theory, it was not until 1964 that John Bell found an error in this formulation and instead proposed an experiment that fully debated the hidden-variables theory [60].

The experiment proposed in fig (4.13) in order to prove the Bell's inequalities considers two observers Alice and Bob. Each of the parties can measure two possible values, Alice's properties are denoted by $a$, and the possible values are $P, Q$, i.e $a \in \{P, Q\}$ and similarly for Bob the properties are denoted by $b$ with possible values $R, S$, i.e $b \in \{R, S\}$. Note that Alice and Bob properties might be the same. In the experiment each measurement of the properties can have two outcomes, this is call a binary outcome, in this case the possible outcomes are $\pm 1$.

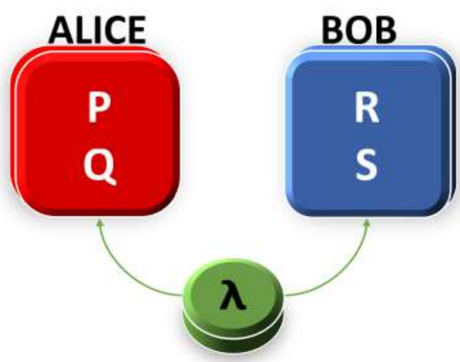


**Figure 4.1:** Proposed experiment to demonstrate the Bell inequality.

Following the logic of Einstein and his collaborators, each one of the parties received a particle prepared by a third party called Charlie.

The correlations between the measured outcomes is given in terms of the properties of each party in the form:

$$E(a, b) = \int_{\Lambda} x(a, b, \lambda) y(a, b, \lambda) P(\lambda|a, b) d\lambda. \tag{4.1}$$

Where $\lambda$ is the hidden-variable, $P(\lambda)$ is the distribution of the hidden-variables, $\Lambda$ is the domain of the hidden-variables, $x$ and $y$ are deterministic functions given $a$, $b$, $\lambda$.

In accordance with the concepts that were previously studied, the following implications are found. Assuming locality the measurements of Alice (Bob) only depend on the observable choice of Alice (Bob), hence:

$$x(a, b, \lambda) = x(a, \lambda),$$

$$y(a, b, \lambda) = y(b, \lambda),$$

also assuming realism is required. This means that the value for the state is determined before it is measured, that is why the distribution is not affected by Alice or/and Bob:

$$P(\lambda|a, b) = P(\lambda).$$

Considering all of this in Eq. (4.1), it is obtained:

$$E(a, b) = \int_{\Lambda} x(a, \lambda) y(b, \lambda) P(\lambda) d\lambda. \tag{4.2}$$

In this way, Clauser-Horne-Shimony-Holt (CHSH) propose the following equation with the particularity that these variables can be measured in the laboratory [61]:

$$CHSH = E(P, R) + E(Q, R) + E(P, S) - E(Q, S). \tag{4.3}$$

Then making the required substitutions:

$$CHSH = \int_{\lambda} [x(P, \lambda)\{y(R, \lambda) + y(S, \lambda)\} + x(Q, \lambda)\{y(R, \lambda) - y(S, \lambda)\}] P(\lambda) d\lambda \tag{4.4}$$

Now using the outcomes' values:

$$-2 \leq [x(P,\lambda)\{y(R,\lambda)+y(S,\lambda)\}+x(Q,\lambda)\{y(R,\lambda)-y(S,\lambda)\}] \leq 2. \tag{4.5}$$

Then,

$$\int_{\Lambda}(-2)P(\lambda)d\lambda \leq CHSH \leq \int_{\Lambda}(2)P(\lambda)d\lambda. \tag{4.6}$$

The representation of the CHSH Bell type inequality, is given by:

$$|CHSH| = |E(P,R)+E(Q,R)+E(P,S)-E(Q,S)| \leq 2. \tag{4.7}$$

This means that the states that satisfy local realism are limited by the value of two.

Now, seeking to corroborate the interpretation of quantum mechanics, an example will be presented with one of the few observable quantities, the Pauli matrices, which in turn have a representation for the Spin (intrinsic angular momentum).

### 4.2.1 Bell inequality for Two-qubit system

One then have a two-qubit system and one is given to Alice and one to Bob, so that the state of this system is:

$$|\psi^{AB}\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A |1\rangle_B - |1\rangle_A |0\rangle_B). \tag{4.8}$$

The observables are defined as:

$$\begin{aligned}
P &\longrightarrow \sigma_z^A,\\
Q &\longrightarrow \frac{1}{\sqrt{2}}(\sigma_z^A + \sigma_x^A),\\
R &\longrightarrow \sigma_z^B,\\
S &\longrightarrow \frac{1}{\sqrt{2}}(\sigma_z^B - \sigma_x^B).
\end{aligned}$$

On evaluating the expectation values using quantum mechanics, for the above operators, it is possible to obtain:

$$\begin{aligned}
E(P,R) &= \langle\psi^{AB}|\, \sigma_z^A \otimes \sigma_z^B \,|\psi^{AB}\rangle = -1,\\
E(Q,R) &= \langle\psi^{AB}|\, \frac{1}{\sqrt{2}}(\sigma_z^A + \sigma_x^A) \otimes \sigma_z^B \,|\psi^{AB}\rangle = \frac{-1}{\sqrt{2}},\\
E(P,S) &= \langle\psi^{AB}|\, \sigma_z^A \otimes \frac{1}{\sqrt{2}}(\sigma_z^B - \sigma_x^B) \,|\psi^{AB}\rangle = \frac{-1}{\sqrt{2}},\\
E(Q,S) &= \langle\psi^{AB}|\, \frac{1}{\sqrt{2}}(\sigma_z^A + \sigma_x^A) \otimes \frac{1}{\sqrt{2}}(\sigma_z^B - \sigma_x^B) \,|\psi^{AB}\rangle = 0.
\end{aligned}$$

Then

$$E(P,R) + E(Q,R) + E(P,S) - E(Q,S) = -2.4142\,. \tag{4.9}$$

The Bell type inequality given in Eq. (4.7) is violated since $|CHSH| > 2$, this result is not consistent with either locality or realism.

This in turn means that quantum phenomena is not consistent with the local hidden-variables theory.

Some time later, in the 80's Aspect et al. [62–65] carried out a series of experiments that confirmed that nature obeys quantum mechanics and not the hidden-variable theory.

Also recently in 2015 [66–68] a series of experiments called loophole-free tests were carried out. These rule out any type of failure in the experimental setup or in the technological conditions required to corroborate the violation of Bell's inequality, which in turn means that nature is consistent with quantum mechanics.

### 4.2.2 Bell inequality for Laguerre Gaussian Beams $LG_0^1$

A very ilustrative application of the violation of the Bell inequality is to function as a witness to quantum entanglement. The Wigner function can be used in order to express a Bell inequality this has been done for Laguerre Gaussian beams [69] and calculations are shown below.
The Wigner representation for LG beams in terms of $\xi = (x, y, Px, Py)$ is:

$$\begin{aligned} W_{m,n}^{LG}(x, y, P_x, P_y) &= (-1)^{m+n} \times \exp(-x^2 - y^2 - P_x^2 - P_y^2) \\ &\times L_m \left[ 2(\frac{1}{2}(x^2 + y^2 + P_x^2 + P_y^2) + (xP_y - yP_x)) \right] \\ &\times L_n \left[ 2(\frac{1}{2}(x^2 + y^2 + P_x^2 + P_y^2) - (xP_y - yP_x)) \right]. \end{aligned} \tag{4.10}$$

Now for the specific case of $m = 1$, $n = 0$ and since $L_0(x) = 1$, $L_1(x) = 1 - x$, in this case the following expression is obtained:

$$L_1 \left[ 2(\frac{1}{2}(x^2 + y^2 + P_x^2 + P_y^2) + (xP_y - yP_x)) \right] = (x + P_y)^2 + (y - P_x)^2 - 1.$$

It is straight forward to arrive at the following final expression for the Wigner's representation:

$$W_{1,0}^{LG}(x, y, P_x, P_y) = [(x + P_y)^2 + (P_x - y)^2 - 1] \exp(-x^2 - y^2 - P_x^2 - P_y^2). \tag{4.11}$$

It is useful to define $\alpha = (x, P_x)$, $\beta = (y, P_y)$ in other to express the Bell inequality in terms of the WF as:

$$B = |W(\alpha, \beta) + W(\alpha, \beta') + W(\alpha', \beta) - W(\alpha', \beta')| < 2. \tag{4.12}$$

To calculate the Bell's inequality for LGB the following considerations will be made, $\alpha = (x = 0, P_x = 0)$, $\alpha' = (x' = x, P_x = 0)$, $\beta = (y = 0, P_y = 0)$ and $\beta' = (y' = 0, P'_y = P_y)$. Calculating all the terms in Eq. (4.12) it is finally obtain:

$$B = \left|(P_y^2 - 1)\exp(-P_y^2) + (x^2 - 1)\exp(-x^2) - ((P_y + x)^2 - 1)\exp(-x^2 - P_y^2) - 1\right|. \tag{4.13}$$

To find the value of the Bell inequality function $B$, given in Eq. (4.13), must be optimized. In other to do this the Nelder Mead Simplex algorithm method was implemented, this method has been widely used to minimize (or maximize) $N$-dimensional functions from an initial polytope.
For this particular case it is straight forward to see in Fig. (4.2) the function is two-dimensional and the value for the maximun violating for the Bell inequality is $|B_{max}| \simeq 2.17$ in $x \simeq \pm 0.45$, $Py \simeq \pm 0.45$, which agrees with the value reported in [69]. Note that in Fig. (4.2) the four quadrants are presented and due to the symmetry of the problem the results are consistent. In reference [69] only the results for one quadrant were reported.

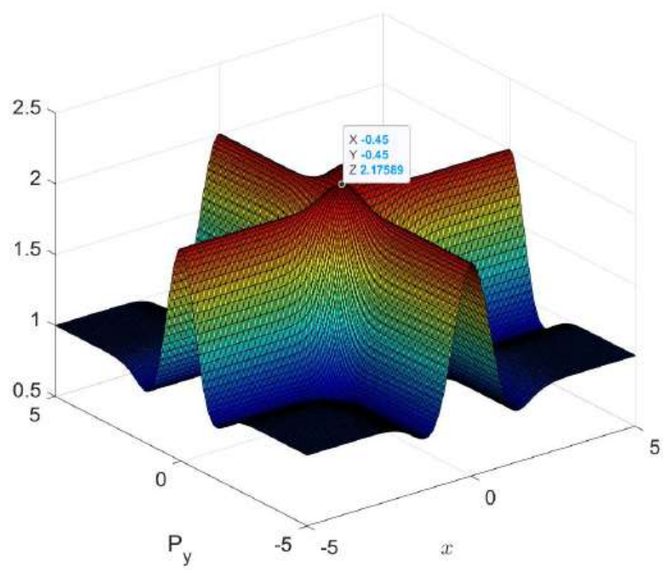


**Figure 4.2:** Bell inequality for $LGB_{1,0}$.

For the generalization for the eight-dimensional space, the variables for the function $B$, given in (4.12), are: $\alpha = (x, P_x)$, $\alpha' = (x', P'_x)$, $\beta = (y, P_y)$ and $\beta' = (y', P'_y)$. Using the same algorithm as before, the maximum value for the 8-$D$ Bell inequality is: $|B_{max}| \simeq 2.24$ for $x \simeq -0.07$, $P_x \simeq 0.05$, $x' \simeq 0.40$, $P'_x \simeq -0.26$, $y \simeq -0.05$, $P_y \simeq -0.07$, $y' \simeq 0.26$, $P'_y \simeq 0.40$. These values also agree with the reported in [69].

Below is a table showing the values of Bell inequality $|B_{max}|$, using the previous procedure, for the families of the $LG_{2,1}$ $LG_{3,1}$ and $LG_{0,5}$.

| $LG_{2,1}$ | $LG_{1,3}$ | $LG_{0,5}$ |
|---|---|---|
| 2.0827 | 2.1452 | 2.2821 |

**Table 4.1:** Bell inequality values for different orders of LG polynomials.

# 5 | Results

In this chapter the results of the Bell inequality for IGB is discussed, as well the Wigner function for these beams, which is a requirement for the Bell type inequality. Also a conceptual discussion about these results is given.

First of all, the methodology used for these results was the following:

- Find the Wigner function for the Ince Gauss beams and compare its limiting cases with known results [56].
- Similarly to [69], propose a Bell CHSH inequality for IGBs, then quantum correlations can be studied.
- Compare previous results with the Laguerre Gauss' Wigner function.

A generalization for Wigner function in SU (2) is proposed in [57]:

$$W_{l,p}(\zeta,\theta,\phi) = \frac{(-1)^N}{\pi^2 \lambda\!\!\!^{-2}} \exp(-Q_0) L_{N-l/2}(Q_0 - 4\mathbf{Q}\cdot\mathbf{u}_r) L_{N+l/2}(Q_0 + 4\mathbf{Q}\cdot\mathbf{u}_r). \tag{5.1}$$

From a mathematical point of view, the Wigner function in the space of the LG and HG beams is invariant trough an intuitive rotation. Based on this, it is proposed that the Wigner function is also invariant for the IG beams, and for this reason, it is the generalization of the entire space and turns out to be Eq. (5.1). In summary, the proposed Wigner function for the IG beams is the generalization of Eq. (5.1).

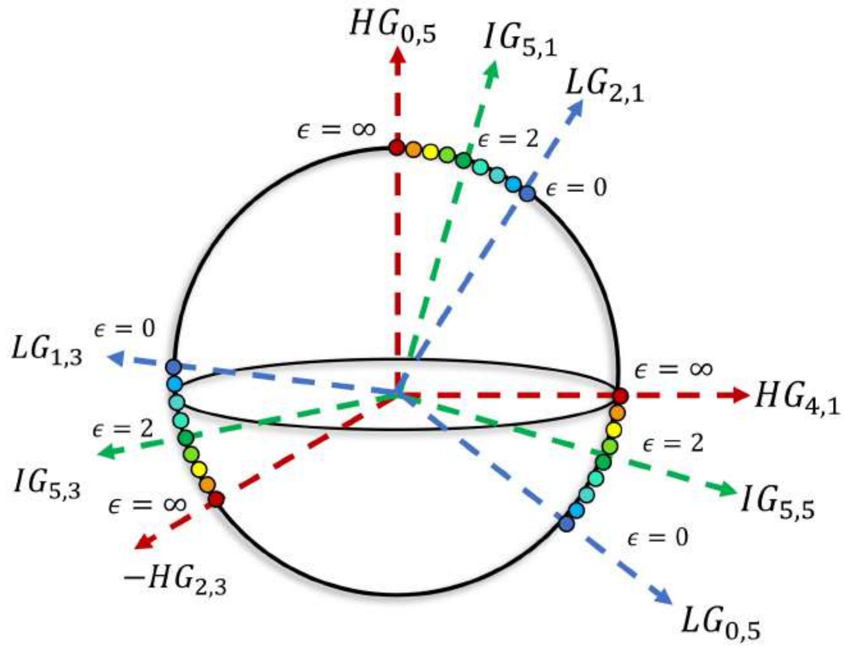


**Figure 5.1:** Geometrical representation in a 3D-sphere for different Gaussian beams in SU(2). Modified figure from [5]

In figure 5.1, a geometrical representation is given, where all the beams that have the same modal number lie in the same sphere. This coincides with the way in which some beams expand into others, this means:

$$IG_{p,m}(\xi, \eta, \epsilon) = \sum_{l,n} D_{l,n} LG_{l,n}(r, \phi) \tag{5.2}$$

$$IG_{p,m}(\xi, \eta, \epsilon) = \sum_{n_x, n:y} C_{n_x,n_y} HG_{n_x,n_y}(x, y) \tag{5.3}$$

Next, the following considerations are taken into account:

- All existing coefficients $D_{l,n}$ and $C_{n_x,n_y}$ are contained in the sphere with modal number $N = 5$, then a way to locate these points in the representation of figure 5.1 is sought.
- This representation allows us to note that all the families of beams (LGB, IGB and HGB) can be decomposed into each other through a change of coordinates within the three-

dimensional vector space.

- As indicated in reference [56], within the subspace being worked on, the Wigner transformation operator has a simple transformation with a rotation between the bases.

Taking all of the above into account, it is then proposed that the Wigner function for the IGBs represents the entire SU(2) space, and that is why Eq. (5.1) may correspond to the Wigner function of Ince Gauss beams. The unit vector $\mathbf{u_r}$ will correspond to the expansion coefficients of the polynomials, for example $\mathbf{D_{n,l}}$ if it is done in the basis of Laguerre Gauss and $\mathbf{C_{nx,ny}}$ if it is done in the basis of Hermite Gauss.

$$\begin{aligned}
W^{IG}_{p,m} &= \exp\left[-2\left(x^2+y^2+P_x^2+P_y^2\right)\right] \qquad (5.4)\\
&\times L_{(p-m)/2,2}\left[Q_0+\left(\left(xy+P_xP_y\right)\cdot n_1+\left(xP_y-yP_x\right)\cdot n_2+\left(x^2-y^2+P_x^2-P_y^2\right)\cdot n_3\right)\right]\\
&\times L_{m,2}\left[Q_0-\left(\left(xy+P_xP_y\right)\cdot n_1+\left(xP_y-yP_x\right)\cdot n_2+\left(x^2-y^2+P_x^2-P_y^2\right)\cdot n_3\right)\right].\\
Q_0 &= \left(x^2+y^2+P_x^2+P_y^2\right).
\end{aligned}$$

Since the coefficients $D_{n,l}$ on the sphere of SU(2) of the same modal number will be used for LGB and IGB, it must be noted that the coordinates associated with the vectors $\mathbf{n} = n1, n2, n3$ will rotate with respect to the coefficients $D_{n,l}$, as it is seen below for $N = 3$:

$$IG_{5,1} = \begin{cases} n_1 = D_{0,5} \\ n_2 = D_{2,1} \\ n_3 = D_{1,3} \end{cases} \qquad IG_{5,3} = \begin{cases} n_1 = D_{2,1} \\ n_2 = D_{1,3} \\ n_3 = D_{0.5} \end{cases} \qquad IG_{5,5} = \begin{cases} n_1 = D_{1,3} \\ n_2 = D_{0,5} \\ n_3 = D_{2,1} \end{cases}$$

When the ellipticity $\epsilon \longrightarrow 0$ the relations LGB's Wigner function are obtained, which is consistent with the limiting cases. This implies that Eq. (5.4) becomes Eq. (A.1). As it is shown above, the Wigner function is required for the formulation of Bell inequality as seen in Eq. (4.12).

## 5.1 Bell inequality for IGB

The following Bell inequality is proposed for the Ince Gauss beams, which takes into account all the previous analysis and on the CHSH inequality.

$$
\begin{gathered}
B = |\exp(-2(x^2+y^2+P_x^2+P_y^2))\times \\
L_{(p-m)/2,2}[(x^2+y^2+P_x^2+P_y^2)+((xy+P_xP_y)\cdot n1+(xP_y-yP_x)\cdot n_2+(x^2-y^2+P_x^2-P_y^2)\cdot n_3)]\times \\
L_{m,2}[x^2+y^2+P_x^2+P_y^2)-((xy+P_xP_y)\cdot n_1+(xP_y-yP_x)\cdot n_2+(x^2-y^2+P_x^2-P_y^2)\cdot n_3)]+ \\
\exp(-2(x'^2+y^2+P_x'^2+P_y^2))\times \\
L_{(p-m)/2,2}[(x'^2+y^2+P_x'^2+P_y^2)+((x'y+P_x'P_y)\cdot n_1+(x'P_y-yP_x')\cdot n_2+(x'^2-y^2+P_x'^2-P_y^2)\cdot n_3)]\times \\
L_{m,2}[x'^2+y^2+P_x'^2+P_y^2)-((x'y+P_x'P_y)\cdot n_1+(x'P_y-yP_x')\cdot n_2+(x'^2-y^2+P_x'^2-P_y^2)\cdot n_3)]+ \\
\exp(-2(x^2+y'^2+P_x^2+P_y'^2))\times \\
L_{(p-m)/2,2}[(x^2+y'^2+P_x^2+P_y'^2)+((xy'+P_xP_y')\cdot n_1+(xP_y'-y'P_x)\cdot n_2+(x^2-y'^2+P_x^2-P_y'^2)\cdot n_3)]\times \\
L_{m,2}[x^2+y'^2+P_x^2+P_y'^2)-((xy'+P_xP_y')\cdot n_1+(xP_y'-y'P_x)\cdot n_2+(x^2-y'^2+P_x^2-P_y'^2)\cdot n_3)]- \\
\exp(-2(x'^2+y'^2+Px'^2+Py'^2))\times \\
L_{(p-m)/2,2}[(x'^2+y'^2+Px'^2+Py'^2)+((x'y'+P_x'P_y')\cdot n1+(x'P_y'-y'P_x')\cdot n_2+(x'^2-y'^2+P_x'^2-P_y'^2)\cdot n_3)]\times \\
L_{m,2}[x'^2+y'^2+P_x'^2+P_y'^2)-((x'y'+P_x'P_y')\cdot n_1+(x'P_y'-y'P_x')\cdot n_2+(x'^2-y'^2+P_x'^2-P_y'^2)\cdot n_3)]|.
\end{gathered}
\tag{5.5}
$$

In fig. (5.2) the results of the Bell inequality are shown, for different values of the ellipticity for the three IG families under consideration, $IG_{5,1}$, $IG_{5,3}$ and $IG_{5,5}$

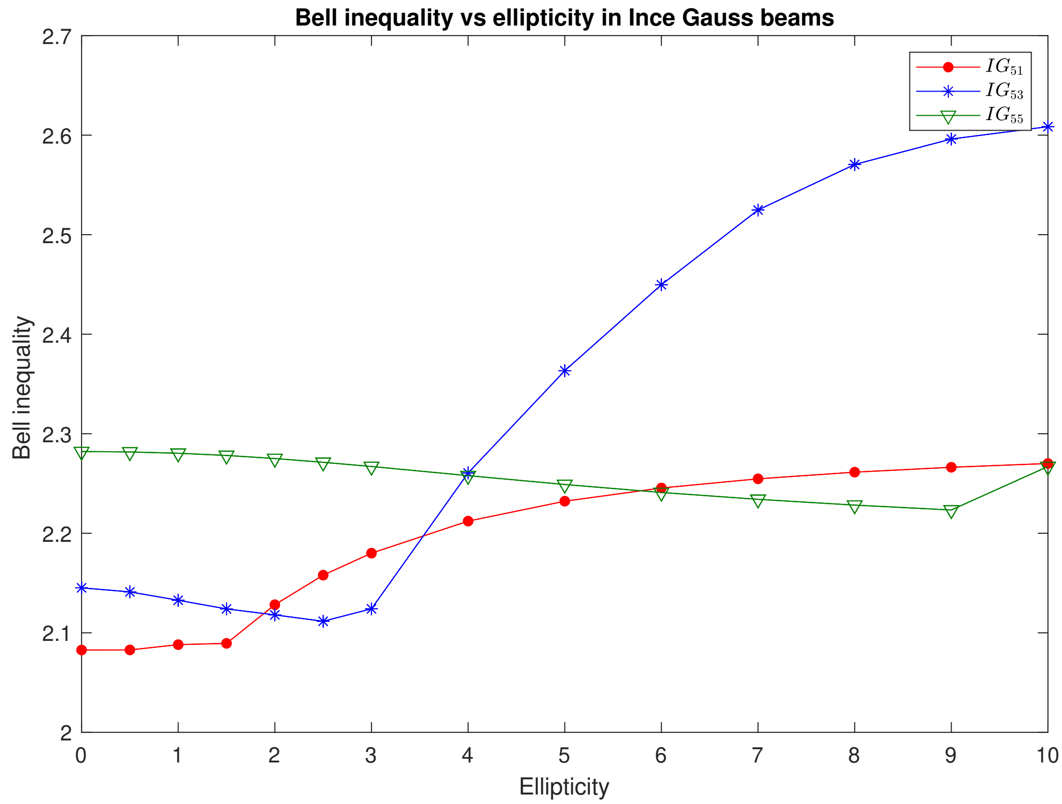


**Figure 5.2:** Bell inequality values for different ellipticity, $\epsilon$. The red line corresponds to $IG_{5,1}$, the blue line represents the $IG_{5,3}$ and green line is $IG_{5,5}$.

## 5.2 Discussion

As far as is known, these results for the Bell inequality for the IGB have not been previously investigated neither from a theoretical or an experimental point of view, therefore at this moment it is not possible to compare the obtained results with any others ones. However, in the limiting case of the ellipticity $\epsilon \to 0$, the obtained results are completely equivalent to the ones for the Laguerre Gaussian beams, this means that the obtained results reproduce in a proper way those ones in the literature. In particular, as it was shown for this limiting case, for both the Wigner function, (A.1) and the Bell inequality, (4.2.2) there is a good agreement.

When the expression for the Wigner function for the IGBs was formulated, a type of rotation between the families of polynomials in SU(2) was taken into account, according to the invariance of the Wigner operator the rotations was presented.

Finally, Eq. (5.5) is proposed as the Bell inequality for IGBs, it is essentially the analytical expression for different values of $m$ and $p$. In Fig. (5.2) the results for different order of IGBs for this Bell inequality as a function of some ellipticity values are shown, some peculiarities of this figure were identified and are listed below:

- There are different crossings, values where the Bell inequality is the same, for different values of the families of the Ince Gauss beams. The families of $IG_{5,1}$ and $IG_{5,3}$ have the crossing at the values of the ellipticity of 2 and 3.5. For the chosen range of values of the ellipticity the $IG_{5,1}$ and $IG_{5,5}$ only have a crossing at the value of 4. While for the families of $IG_{5,5}$ and $IG_{5,1}$ the crossing occurs at the values of the ellipticity of 6, and 10. It also should be noted that for these families of IG beams, and for the chosen range of values of the ellipticity, it is possible to show a violation of the Bell inequality.

- The curves for $IG_{5,3}$ and $IG_{5,5}$ show a decreasing behaviour, even a linear one, up to a certain value for the ellipticity, which is a different one for the two curves, this values is an inflexion point, since after these values the curves show an increasing behavior. However for the family of $IG_{5,1}$ the curve, for the chosen range of values of the ellipticity, shows a monotonically increasing behaviour. For $IG_{5,1}$ and $IG_{5,3}$, it is evident that after a certain value, for $IG_{5,1}$ and $IG_{5,3}$, it is evident that after a certain value, these are nonlinear curves.

- For $IG_{5,1}$ and $IG_{5,3}$, as the value of the ellipticity increases, the higher the order of the structured beams, the higher the violation of the Bell inequality is.

- Since, it was necessary to make an expansion for the LGB in order to obtain Eq. (5.5), it is not possible to perfor a broader analysis for this figure for larger values of *epsilon*, since these would belong to a different regime.

# 6 | Conclusions

The research on how powerful the applications of structured light are in quantum optics and quantum information is highlighted in this thesis. Bell-type inequalities are important, since they are a form to show non-locality properties of a system, like structured beams. Therefore it is important to propose Bell-type inequalities which can be analyzed, in particular the CHSH inequality is feasible to perform in an experiment. Violations of the Bell-type inequalities for structured beams demonstrate their nonlocal properties as well as it is a form to show entanglement.

In this thesis, the Bell inequality was studied, obtaining an analytical expression for the IGBs, in this process an expression for the Wigner function was also investigated. These two expressions had not been previously studied in the literature, even from a theoretically point of view. In this thesis, expressions for both the Wigner function and the Bell inequality are proposed.

Since it was a theoretical study, this provided the possibility to make the comparison with other functions of Wigner and Bell inequalities, for example, these values for LGBs. It was found that the values coincide completely with previous ones reported in the literature for comparison with this limit case where $\epsilon = 0$.

Both in the proposed expression and in Fig.(5.2) it is clearly seen that for the three orders of

IGBs Bell inequality is violated, which means that it gives a solid criterion to show quantum entanglement for this type of structured light.

Due to the similarities between the representations of the Poincaré sphere for the SU(2) group, a change of basis between IGBs, LGBs and HGBs by means of the coefficients $D_n, l$, was used to express the rotations between the families of structured beams.

During this research, many techniques were used in order to propose an expression, either of the Wigner function or the Bell inequality itself. Some of these techniques are shown in the Appendices. It is worth to mention that through group theory methods it might be possible to get a solid mathematical justification of the expression proposed here. However this is beyond the scope of the present master's thesis, since it requires more time to analyse.

# A | Appendix A

## Wigner function for LGB in cylindrical coordinates

Using the transformation between polar and cartesian coordinates: $x = q\cos\theta$, $y = q\sin\theta$, $P_x = p\cos\phi$ and $P_y = p\sin\phi$ together with the relations given in Eq. (3.34) for $Q_0$ and $Q_2$, the Wigner function for the LGB in polar coordinates is:

$$W^{LG}_{j,m} = L_{j+m}\left[2\left(\frac{q^2}{w^2} + \frac{w^2}{4\lambda}p^2 + \frac{pq}{\lambda}\sin(\theta-\phi)\right)\right] L_{j-m}\left[2\left(\frac{q^2}{w^2} + \frac{w^2}{4\lambda}p^2 - \frac{pq}{\lambda}\sin(\theta-\phi)\right)\right] \exp\left(\frac{q^2}{w^2} + \frac{w^2}{4\lambda}p^2\right). \tag{A.1}$$

In fig. (A.1) the Wigner function $W^{LG}_{j,m}(p,q,\delta)$ given in Eq. (A.1) is visualized. Columns show the WF for a fixed value of $j$ and $m$, varying the value of the angle $\delta = \theta - \phi$, while rows show the WF for fixed value of $\delta$ varying the polynomial's order. Note that the $j - m \geq 0$ restriction is imposed since negative values for the orders of the Laguerre polynomials are not allowed, likewise, for the Wigner function are not too. Also one can notice that the WF, see Fig. (A.1), has negative values. This is an evidence for non-classical states. Additionally take note that the shape of the WF is periodic following the angle's change.

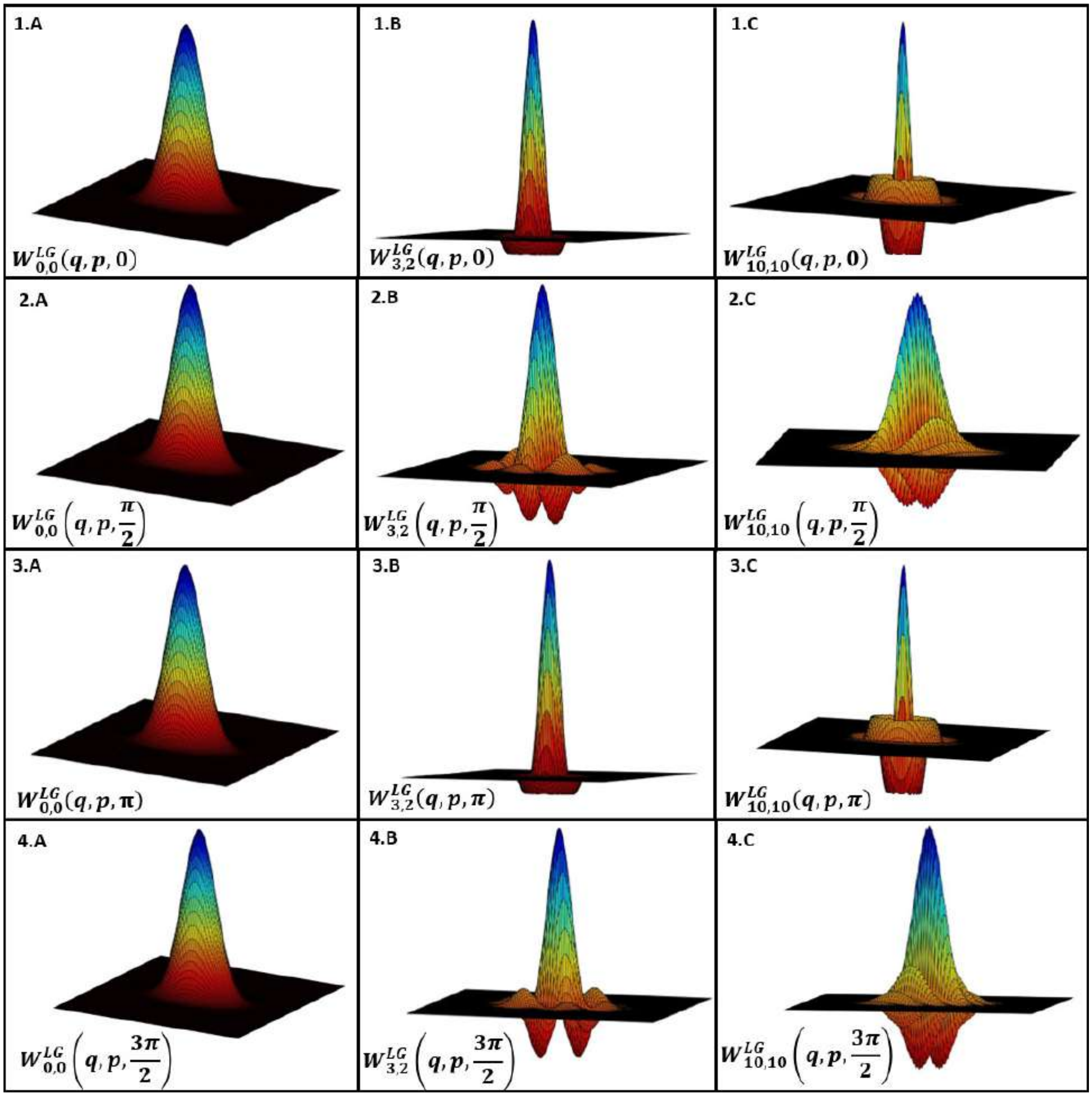


**Figure A.1:** Wigner function for Laguerre Gaussian beams, in the columns the angle is changed, and in the rows the polynomial's order is varying.

# B | Appendix B

## Momentum operator in elliptic coordinates

If one wish to write an analytic expression for the Wigner function in elliptical coordinates it is necessary to express the momentum operator in this coordinates. However, a representation of this operator in elliptical coordinates was not found in the literature, therefore for this research it was necessary to perform calculations to find an analytical expression, which is shown below. In order to perform a change of variables, the scale factors for elliptical coordinates are evaluated from $\vec{r} = (f\cosh(\xi)\cos(\eta), f\sinh(\xi)\sin(\eta), z)$ obtaining

$$h_\eta = h_\xi = \sqrt{\cosh^2(\xi) - \cos^2(\eta)}. \tag{B.1}$$

From analytical mechanics it is known that:

$$p_i = -i\hbar \frac{1}{f(q)} \frac{\partial}{\partial q_i} f(q), \tag{B.2}$$

when $p_i$ is momentum and $q_i$ generalized coordinate, also $f(q)$ is a arbitrary function. The first part of this process is to find the surface elements of each coordinate, this means:

$$dS_\eta = f\sqrt{(\cosh^2\xi - \cos^2\eta}d\xi dz$$
$$dS_\xi = f\sqrt{(\cosh^2\xi - \cos^2\eta}d\xi dz$$
$$dS_z = f^2(\cosh^2\xi - \cos^2\eta)d\xi d\eta$$

To find $p_\eta$, $f(\xi,\eta,z) = 1$ is assumed and the expected value is calculated so that:

$$< p'_\eta >= -i\hbar\int_0^{2\pi}\int_{-\infty}^{\infty}\left[\int_0^{\infty} f\left(\sqrt{\cosh^2\xi - \cos^2\eta}\right)\Psi^*(\vec{r},t)\frac{\partial\Psi^*(\vec{r},t)}{\partial\eta}d\eta\right], \tag{B.3}$$

after some manipulation in the integral, is possible see that:

$$p_\eta = p'_\eta + i\hbar\left(\frac{2\cos(\eta)\sin(\eta)}{f\sqrt{\cosh^2\xi - \cos^2\eta}}\right) \tag{B.4}$$
$$p_\eta = -i\hbar\left(\frac{\partial}{\partial\eta} + \frac{\sin(2\eta)}{f\sqrt{\cosh^2\xi - \cos^2\eta}}\right).$$

Due to the respective surface differential and scale factor are the same for $\eta$ and $\xi$ coordinates, the procedure to find the momentium operator is the same, this means that:

$$p_\xi = -i\hbar\left(\frac{\partial}{\partial\xi} + \frac{\sinh(2\xi)}{f\sqrt{\cosh^2\xi - \cos^2\eta}}\right). \tag{B.5}$$

With a similar procedure, taking into account the respective line differential and with the fact that $\frac{\partial dS_z}{\partial z} = 0$, the operating momentum for the $z$ coordinate is:

$$p_z = -i\hbar\frac{\partial}{\partial z}. \tag{B.6}$$

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