Spectral analysis of phase shifting algorithms

M. Servin,[1,*], J. C. Estrada[1], and J. A. Quiroga[2]

[1]Centro de Investigaciones en Óptica A. C., Loma del Bosque 115,
Col. Lomas del Campestre, 37150, León Guanajuato, México
[2]Departamento de Óptica, Univ. Complutense de Madrid, Ciudad Universitaria s/n,
28040 Madrid, Spain.
[*]mservin@cio.mx

Abstract: Systematic spectral analysis of Phase Shifting Interferometry (PSI) algorithms was first proposed in 1990 by Freischlad and Koliopoulos (F&K). This analysis was proposed with the intention that "in a glance" the main properties of the PSI algorithms would be highlighted. However a major drawback of the F&K spectral analysis is that it changes when the PSI algorithm is rotated or its reference signal is time-shifted. In other words, the F&K spectral plot is different when the PSI algorithm is rotated or its reference is time-shifted. However, it is well known that these simple operations do not alter the basic phase demodulation properties of PSI algorithms, except for an unimportant piston. Here we propose a new way to analyze the spectra of PSI algorithms which is invariant to rotation and/or reference time-shift among other advantages over the nowadays standard PSI spectral analysis by F&K.

OCIS codes: (120.3180) Interferometry; (120.2650) Fringe Analysis.

References and links

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1. Introduction

Spectral analysis of PSI algorithms was not systematically addressed before the paper of Freischlad and Koliopoulos (F&K) [1]. F&K have correctly interpreted that PSI algorithms are actually complex quadrature filters in the temporal domain [1] (see also [2,3]) and used the Fourier transform to find the spectra of these filters. In addition to this, they have proposed a spectral analysis in order to graphically interpret the main features of these quadrature filters (PSI algorithms). However the main drawback of this spectral analysis is that it changes when the PSI filter is rotated in the complex plane and/or its reference is time-shifted. This analysis changes so much with rotation that one may have the false impression of dealing with a different PSI algorithm. However it is well known that the estimated phase of a rotated or time-shifted PSI algorithm remains unchanged, except for an unimportant piston phase [4,5].

Here we propose a new way of to analyze the spectra of PSI algorithms based on the frequency transfer function [6] of the complex filter associated with the PSI algorithm. This

PSI filter's spectrum is *invariant to rotation and/or reference time-shift*. Also this new spectral representation has zero response at frequencies rejected by the PSI quadrature filter.

2. The aim of a quadrature filter in phase shifting interferometry

Let us begin by showing the usual mathematical model of a temporal interferometric signal as,

$$I(x, y, t) = a(x, y) + b(x, y) \cos[\varphi(x, y) + \omega_0 t]. \tag{1}$$

Where $a(x,y)$, $b(x,y)$, and $\phi(x,y)$ are the background, contrast and the searched phase. Finally ω_0 is the temporal carrier (in radians) of the phase shifted interferograms. Rewriting it as,

$$I(x, y, t) = a + \tfrac{b}{2} \exp[-i(\varphi + \omega_0 t)] + \tfrac{b}{2} \exp[i(\varphi + \omega_0 t)]. \tag{2}$$

The spatial dependence of the functions $a(x,y)$, $b(x,y)$ and $\phi(x,y)$ were omitted for clarity. The aim of a PSI algorithm is to filter out $a(x,y)$, and *one* of the two complex exponentials. Therefore the output signal $Ic(t)$ is,

$$Ic(t) = [b(x, y) / 2] \exp\{ -i[\varphi(x, y) + \omega_0 t]\} = h(t) * I(x, y, t). \tag{3}$$

Where the symbol * denotes a one-dimensional convolution, and $h(t)$ is the PSI filter's complex (quadrature) impulse response,

$$h(t) = hr(t) + i\, hi(t). \tag{4}$$

Denoting by $H(\omega) = F[h(t)]$ its Fourier transform. The minimal conditions on the frequency transfer function $H(\omega)$ to obtain the complex signal $Ic(t)$ in Eq. (3) from the interferogram are:

$$H(-\omega_0) \neq 0, \quad \text{and} \quad H(\omega_0) = H(0) = 0. \tag{5}$$

These conditions on $H(\omega)$ are not normally used in the field of few-steps PSI interferometry. Instead due to [1], one normally uses the following equivalent set of requirements,

$$\left\{ F[hr(t)] - i\, F[hi(t)] \right\}\Big|_{\omega=-\omega_0} = 0, \quad \text{and} \quad F[hr(t)]\big|_{\omega=0} = F[hi(t)]\big|_{\omega=0} = 0. \tag{6}$$

3. Rotation and reference time-shift of PSI algorithms

In this section we discuss the rotation and reference time-shift of PSI algorithms. Malacara *et. al*. [4], page 239, and Schmit *et. al*. [5] have discussed this before. As it is well known [1], a PSI algorithm (or quadrature filter) may be expressed in terms of the real $hr(t)$ and imaginary $hi(t)$ parts of a complex impulse response filter $h(t) = hr(t) + i\, hi(t)$ as,

$$\tan[\varphi(x, y)] = \frac{hi(t) * I(x, y, t)}{hr(t) * I(x, y, t)}\Bigg|_{t=0}. \tag{7}$$

The rotation (by Δ_0) and the reference time-shift (t-t_0) of this complex linear filter are formally represented by,

$$h(t) \exp[i\Delta_0], \quad \text{and} \quad h(t - t_0). \tag{8}$$

The most general impulse response of a linear-quadrature digital-filter (PSI algorithm) is, $\exp[i\omega_0 t]\left\{ \sum_n a_n \delta(t - nT) \right\}$ where $\exp[iw_0 t]$ is the reference signal (local oscillator), T is the sampling rate, and the coefficients a_n are weighting (possible complex) constants. Time shifting the reference one obtains,

$$h(t - t_0) = \exp[i\, \omega_0(t - t_0)]\left\{ \sum_n a_n \delta(t - nT) \right\} = h(t) \exp[-i\omega_0 t_0]. \tag{9}$$

Therefore, time-shifting the reference signal also implies filter's rotation, with $\Delta_0 = -\omega_0 t_0$. This was first observed in Malacara *et. al.* [4], page 239. Rotating $h(t)$ by Δ_0 one obtains,

$$h(t)\exp[i\,\Delta_0] = hr(t)\cos(\Delta_0) - hi(t)\sin(\Delta_0) + i\,[hr(t)\sin(\Delta_0) + hi(t)\cos(\Delta_0)]\,. \quad (10)$$

As Eq. (10) shows, rotation and/or reference time-shift linearly mix up the real and imaginary parts of the original PSI filter $h(t)$. This give apparently "new" PSI algorithms that looks quite different albeit being the same one [4,5]. The family of PSI algorithms obtained from the rotated (or time shifted $\Delta_0 = -\omega_0 t_0$) quadrature filter according to Eq. (10) looks now as,

$$\tan[\varphi(x,y) + \Delta_0] = \left.\frac{[hr(t)\sin(\Delta_0) + hi(t)\cos(\Delta_0)]*I(t)}{[hr(t)\cos(\Delta_0) - hi(t)\sin(\Delta_0)]*I(t)}\right|_{t=0}\,. \quad (11)$$

4. The Freischlad and Koliopoulos spectral analysis

The spectral analysis of PSI algorithms published by Freishlad *et. al.* [1] consist on finding the Fourier transform of the real and imaginary parts of $h(t)$, *separately*, that is

$$\begin{aligned} Hi(\omega) &= F[hi(t)] \\ Hr(\omega) &= F[hr(t)] \quad. \end{aligned} \quad (12)$$

Having these Fourier transforms, F&K recommend to analyze these spectra *without constant or common phase factors* [1], and in this way obtain two real functions to plot.

But as we said, if we rotate the PSI algorithm one may easily run into trouble, because now the rotated spectra have the following form,

$$\begin{aligned} Hi(\omega) &= F[hr(t)\sin(\Delta_0) + hi(t)\cos(\Delta_0)] \\ Hr(\omega) &= F[hr(t)\cos(\Delta_0) - hi(t)\sin(\Delta_0)] \quad. \end{aligned} \quad (13)$$

The spectral analysis that results from this rotated PSI algorithm is clearly different to the one in Eq. (12), and may give the false impression that they correspond to two different PSI algorithms. In other words, *the F&K spectral representation would give different spectral plots for each rotation of the same PSI algorithm*. In our herein presented spectral representation, this drawback does not exist.

5. Our proposed spectral analysis

As revised in the previous section, the spectral analysis in [1] take the Fourier transform of the real $hr(t)$ and the imaginary $hi(t)$ components of $h(t)$ *separately*; Eq. (12). This is followed by a rule to obtain two real functions to plot and analyze. On the other hand, we propose to analyze the spectra of the complex sum $h(t) = hr(t) + i\,hi(t)$. So we only take the Fourier transform of $h(t)$, *i.e.* $H(\omega) = F[h(t)]$. This new way of analyzing the spectra of PSI algorithms have (as shown here) interesting and useful consequences.

It is well known that the spectra $H(\omega)$ [6] is the ratio of the output's signal spectra $Ic(\omega)$ to the input's signal spectra $I(\omega)$,

$$H(\omega) = |H(\omega)|\exp[i\,\Delta(\omega)] = \frac{Ic(\omega)}{I(\omega)}\,. \quad (14)$$

Where the function $|H(\omega)|$ is the magnitude of $H(\omega)$, and $\Delta(\omega)$ is the phase introduced by the linear filter. Moreover, the filter's phase $\Delta(\omega)$ is constant in many popular PSI algorithms [4]. The output of any PSI filter is the complex analytical signal $Ic(t)$ in Eq. (3) associated with the interferogram's real signal $I(t)$. Therefore, as it is shown in Fig. 1 we may have an infinite number of transfer functions $H(\omega)$ that comply with Eq. (5). In Fig. 1 we show two block diagrams of linear quadrature PSI filters that may be used to obtain our searched analytical signal $Ic(t)$ at $-\omega_0$ from the real interferometric signal $I(t)$.
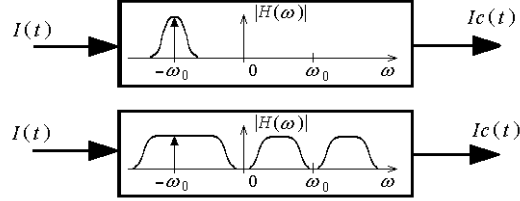
Fig. 1. Spectral magnitude $|H(\omega)|$ of two PSI algorithms. We show the input interferogram as $I(t)$, and the output complex signal as $Ic(t)$. We can clearly see that these two filters pass only the left side analytical signal at frequency $-\omega_0$, while rejecting the background $a(x,y)$ at $\omega = 0$ and the right side analytical signal at $+\omega_0$.

As we said, most PSI algorithms published to date have a constant phase shift $\Delta(\omega) = \Delta_0$ [4], therefore in these cases the plot of $|H(\omega)|$ may be used as its spectrum, which is given by

$$\left|H(\omega)\right| = \left|F[h(t)]\right| = \sqrt{Re(\omega)^2 + Im(\omega)^2} \ . \tag{15}$$

Where $H(\omega) = Re(\omega) + i\ Im(\omega)$, being $Re(\omega)$ and $Im(\omega)$ real functions of ω. As seen in the next section, using this new way of plotting the PSI spectra, one can easily visualize the main properties of a PSI algorithm. Moreover, it is well known that [6]; the rotated $h(t)\exp[i\Delta_0]$; the time-shifted $h(t-t_0)$, and the original PSI filter $h(t)$ have all the same magnitude $|H(\omega)|$,

$$\left|H(\omega)\right| = \left|F[h(t)]\right| = \left|F\{h(t)\exp[i\Delta_0]\}\right| = \left|F[h(t-t_0)]\right| . \tag{16}$$

6. Examples

In this section, using two examples, we illustrate the rotational/time-shifting ambiguity of the F&K analysis and how our new PSI spectral analysis based on $|H(\omega)|$ avoids them.

6.1 Two four-step PSI algorithms

Let us begin with two popular 4-step PSI algorithms [4],

$$\tan[\varphi_1(x,y)] = \frac{I_1 + I_2 - I_3 - I_4}{I_1 - I_2 - I_3 + I_4}, \quad \text{and} \quad \tan[\varphi_2(x,y)] = \frac{I_2 - I_4}{I_1 - I_3}. \tag{17}$$

The phase shift among the interferograms used in these PSI algorithms is 90° degrees. In the first case the actual angles may be: $-3\pi/4$, $-\pi/4$, $\pi/4$, and $3\pi/4$ radians; while in the second case: $-\pi/2$, 0, $\pi/2$, and π radians. Denoting by $h1(t) = hr1(t) + i\ hi1(t)$ and $h2(t) = hr2(t) + i\ hi2(t)$, the two impulse responses associated with these PSI algorithms are:

$$h1(t) = \delta(t + 3\pi/4) - \delta(t + \pi/4) - \delta(t - \pi/4) + \delta(t - 3\pi/4) +$$
$$i\left[\delta(t + 3\pi/4) + \delta(t + \pi/4) - \delta(t - \pi/4) - \delta(t - 3\pi/4)\right] \tag{18}$$
$$h2(t) = \delta(t - \pi/2) - \delta(t + \pi/2) + i\left[\delta(t) - \delta(t + \pi)\right] \ .$$

By looking at these impulse responses, and according to F&K [1], one would need to analyze the following spectra,

$$Hr1(\omega) = 2\cos(\pi\omega/4) - 2\cos(3\pi\omega/4), \quad Hi1(\omega) = 2i\left[\sin(\pi\omega/4) + \sin(3\pi\omega/4)\right]$$
$$Hr2(\omega) = -2i\sin(\pi\omega/2) \quad , \qquad\qquad Hi2(\omega) = 1 - \exp(i\ \pi\omega) \quad . \tag{19}$$

Because of the remarkable difference between the Fourier transform of the components of these two filters, one may wrongly think that we are dealing with two different 4-step PSI algorithms. However using Eq. (15), one can easily find the magnitude of the Fourier transform of $h1(t)$ and $h2(t)$ and realize that both are proportional,

$$|H1(\omega)| = \sqrt{2} \, |H2(\omega)|. \tag{20}$$

The plot of $|H1(\omega)|$ and $|H2(\omega)|$ are shown in Fig. 2. We can see that in this representation both filters have proportional spectra and therefore (except for a piston phase equals to $\pi/4$ introduced by $H2(\omega)$ [5]) *have identical phase demodulating properties.*
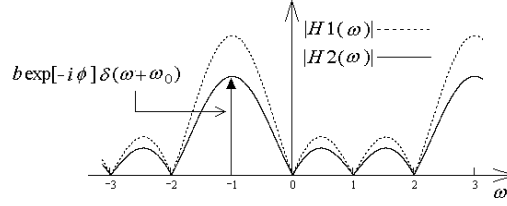

Fig. 2. Spectral magnitude of the two 4-step PSI algorithms in Eq. (17). We see that they are actually proportional and as a consequence, have identical phase demodulation properties. These PSI filters let pass the left side analytical signal at $\omega = -1.0$ while rejecting the background at $\omega = 0$ and the complex signal at $\omega = 1.0$.

6.2 Five-step Schwider-Hariharan PSI algorithm

Let us take another popular PSI algorithm; the Schwider-Hariharan [4] 5-step one, which is,

$$\tan[\varphi(x, y)] = \frac{2(I_2 - I_4)}{I_1 - 2I_3 + I_5}. \tag{21}$$

Where the phase steps are $-\pi$, $-\pi/2$, 0, $\pi/2$, and π. This PSI algorithm has the following associated impulse response $h(t)$,

$$h(t) = \delta(t - \pi) - 2\delta(t) + \delta(t + \pi) + i\left[2\delta(t - \pi/2) - 2\delta(t + \pi/2)\right]. \tag{22}$$

According to [1] one would need to plot and analyze the following spectra,

$$\begin{aligned} Hr(\omega) &= F[\delta(t - \pi) - 2\delta(t) + \delta(t + \pi)] = -2 + 2\cos(\pi\omega) \\ Hi(\omega) &= F[2\delta(t - \pi/2) - 2\delta(t + \pi/2)] = -4i\sin(\pi\omega/2) \end{aligned} \tag{23}$$

However if we rotate $h(t)$ by Δ_0 radians we obtain,

$$h(t)\exp[i\,\Delta_0] = \left\{\delta(t - \pi) - 2\delta(t) + \delta(t + \pi) + i\left[2\delta(t - \pi/2) - 2\delta(t + \pi/2)\right]\right\}\exp[i\,\Delta_0]. \tag{24}$$

If the rotation angle is (for example) $\Delta_0 = \pi/4$ then the rotated impulse response becomes,

$$\begin{aligned} \sqrt{2}\ h(t)\exp[i\,\pi/4] = &\ \delta(t - \pi) + 2\delta(t - \pi/2) - 2\delta(t) - 2\delta(t + \pi/2) + \delta(t + \pi) + \\ &\ i\,[\delta(t - \pi) - 2\delta(t - \pi/2) - 2\delta(t) + 2\delta(t + \pi/2) + \delta(t + \pi)]. \end{aligned} \tag{25}$$

Giving the following rotated 5-step PSI algorithm (this equivalence was first observed in [5]),

$$\tan[\varphi(x, y) + \pi/4] = \frac{I_1 + 2I_2 - 2I_3 - 2I_4 + I_5}{I_1 - 2I_2 - 2I_3 + 2I_4 + I_5}. \tag{26}$$

The Fourier transform of the real and imaginary parts of this rotated 5-step algorithm are now,

$$\begin{aligned} \sqrt{2}\ Hr(\omega) &= -2 + 2\cos(\pi\omega) - 4\,i\,\sin(\pi\omega/2) \quad, \\ \sqrt{2}\ Hi(\omega) &= 2 - 2\cos(\pi\omega) - 4\,i\,\sin(\pi\omega/2) \quad. \end{aligned} \tag{27}$$

Given the big difference between Eq. (23) and Eq. (27), and according to the analysis proposed in [1] one may easily think that we are dealing with two different PSI algorithms!. However it is easy to see that the magnitude of $H(\omega) = F[h(t)]$ of both algorithms are (of course) identical

$$\left|H(\omega)\right| = \left|-2 + 2\cos(\pi\omega) + 4\sin(\pi\omega/2)\right|. \qquad (28)$$

This is shown in Fig. 3, where we can see that this 5-step PSI filter will eliminate the complex exponential at $\omega = 1.0$ and the background of the interferogram $I(t)$ at $\omega = 0$.
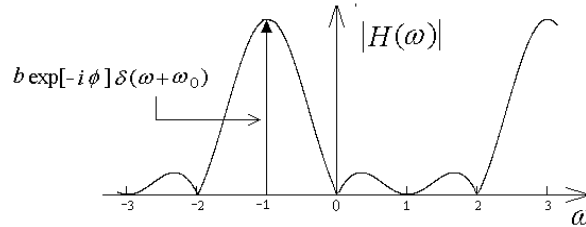
Fig. 3. Spectral magnitude |$H(\omega)$| of the 5-step Schwider-Hariharan algorithm and its rotated twin. Both are of course identical. Only the left side analytical signal at $\omega = -1.0$ passes through, while rejecting the background and the complex signal at $\omega = +1.0$.

6.3 Discusion

We now list some advantages of our new way of analyzing the spectra of PSI algorithms.

1. The magnitude of the frequency response of the filter is invariant to the PSI filter rotations and/or constant time-shift of the reference signal (local oscillator).

2. The signals at frequencies $\omega_1, \ldots, \omega_n$ that the PSI filter rejects are clearly seen as zeroes over the frequency axis *i.e.* |$H(\omega_1)$| = … = |$H(\omega_n)$| = 0.

3. The properties of the PSI algorithms in the neighborhood of the rejected frequencies are also clearly shown. For example the detuning robustness of the Schwider-Hariharan 5-step algorithm is shown as a zero for the first derivative of $H(\omega)$ at $\omega = 1.0$.

4. In [7] we show that the phase noise in a PSI algorithm is proportional to the integral of |$H(\omega)$|^2. So at a glance one may estimate the noise rejection of two "competing" PSI algorithms by their area under |$H(\omega)$|^2 for the same output signal's energy.

7. Conclusions

In this paper we have proposed a new way to analyze the spectra of PSI quadrature filters which is invariant to PSI algorithm rotation or reference time-shift. As it is well known [4,5] a simple rotation and/or time-shift of a PSI algorithm do not alter the PSI phase but for an irrelevant piston. However, as simple as these operations are, according to the analysis proposed in [1] it gives different spectral plots, giving the false impression that the PSI algorithm have somehow changed. Additionally our new PSI spectral analysis clearly shows as zeroes in $H(\omega)$ the rejected frequencies, and no need for further interpretation is required. On top of this, the spectral analysis based on $H(\omega)$ is the most usual (and intuitive) manner to graphically represent the spectra of any optical and/or electrical filter in engineering.

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