Degree of polarization as a criterion to obtain the nine bilinear constraints between the Mueller–Jones matrix elements

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The degree of polarization is employed as a criterion to find the nine independent relations among the elements of the Mueller–Jones matrix. This procedure is applied by considering a previously determined, physically realizable Mueller matrix. On the other hand, the nine bilinear constrains are obtained by directly measuring the degree of polarization from an outgoing beam of light from an optical system by considering nine incident states of light taken from the Poincaré sphere. For practical purposes, all the incident polarization states must be scanned from the Poincaré sphere in order to satisfy the overpolarization and the overgain conditions, respectively, for the physical realizability of the Mueller matrix. © 2007 Optical Society of America

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1. Introduction

In past years a great interest has been directed to the Mueller matrix formalism, in both theoretical and applicability fields. The Mueller matrix associated with a general optical system can be determined by using different techniques, which can be classified as modulated and nonmodulated, respectively. Some of the modulated techniques can be consulted in well known and respectable references [1–3], among many others. As a matter of fact, a lot of commercial equipment, modulated or not, ensures that for the Stokes parameters or the Mueller matrix elements, complete measurement can take a few seconds or less [4]. For a dynamical system under study, this could be a restriction and not an advantage. Maybe the possibility of fixing the time-average measurement for each element, according to the users' needs, could be a solution for practically any type of systems, dynamic or static, because the polarization average is usually a required relevant parameter. On the other hand, the simplest technique, the ideal polarimetric arrangement, uses classical optical elements such as linear polarizers and quarter-wave plates for the polarization-state generator (PSG) and the polarization-state detector (PSD) or analyzer, respectively. With that arrangement both PSG and PSD setups are manually controlled, and the classical optical elements responses are considered as ideal [5–12]. By using this arrangement, the complete Mueller matrix can be determined for a general optical system by using 49, 36, or 16 irradiance (intensity) measurements, respectively [7–12]. Indeed, at least two works have been reported where totally unpolarized light has been used as a PSG entrance and an aperture (open hole) has been employed as the corresponding PSD [9,12]. For one-dimensional rough surfaces, the Mueller matrix has been determined by using six and four intensity measurements, respectively [7,13,14]. With respect to an orthogonal Cartesian coordinate system, a one-dimensional rough surface is a surface whose profile varies along the x axis and remains constant along the y axis.

In this work, the term depolarization refers to the loss in the degree of polarization as the light is scattered or is propagated through an optical system. The mathematical foundation of the Mueller matrices is broad, and many references can be found in recent

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texts that make a remarkable emphasis on the Mueller–Stokes matricial formalism [2,3,15]. In general, the physically realizable conditions for Mueller matrix elements have a practical importance for calibration of polarimetric instruments and estimation or errors. One of the physically realizable conditions for Mueller matrices has been reported by Jones, who has established that [16] "if the optical system does not depolarize, then 9 identities exist among the 16 Mueller coefficients, so that only 7 of them are independent." In this case the Mueller matrix can be derived from a Jones matrix, and it has been called the Mueller–Jones matrix. Some years later, van de Hulst asserted that there exist nine independent relationships between the 16 elements of the Mueller–Jones matrix [17]. However, none of them provided an explicit derivation of this statement. At the beginning of the 1980s, Barakat [18] and Fry and Kattawar [19] derived the explicit nine bilinear constraints between the 16 elements of the Mueller–Jones matrices. Maybe the most complete study of the algebraic relationships existing between the 16 elements of the Mueller–Jones matrices has been reported by Hovenier, van de Hulst, and van der Mee [20] and Hovenier [21]. By dealing with 120 possible products between distinct elements of the Mueller matrix, they have found 30 equations, each containing four terms, which are products of two elements, with no repeating products. Even when Hovenier and coauthors have based their analysis in the scattering of light by a single particle, their results are valid for nondepolarizing systems also. They also have derived the nine bilinear constraints by using nine specific polarization states, assuming the scattering of a fully polarized wave by a single particle results in a completely polarized beam. The nine bilinear constraints are given by the following equations [18–21]:

$$m_{01}{}^2 - m_{11}{}^2 - m_{21}{}^2 - m_{31}{}^2 + m_{00}{}^2 - m_{10}{}^2 - m_{20}{}^2 - m_{30}{}^2 = 0, \tag{1}$$

$$m_{02}{}^2 - m_{12}{}^2 - m_{22}{}^2 - m_{32}{}^2 + m_{00}{}^2 - m_{10}{}^2 - m_{20}{}^2 - m_{30}{}^2 = 0, \tag{2}$$

$$m_{03}{}^2 - m_{13}{}^2 - m_{23}{}^2 - m_{33}{}^2 + m_{00}{}^2 - m_{10}{}^2 - m_{20}{}^2 - m_{30}{}^2 = 0, \tag{3}$$

$$m_{00}m_{01} - m_{10}m_{11} - m_{20}m_{21} - m_{30}m_{31} = 0, \tag{4}$$

$$m_{00}m_{02} - m_{10}m_{12} - m_{20}m_{22} - m_{30}m_{32} = 0, \tag{5}$$

$$m_{00}m_{03} - m_{10}m_{13} - m_{20}m_{23} - m_{30}m_{33} = 0, \tag{6}$$

$$m_{01}m_{02} - m_{11}m_{12} - m_{21}m_{22} - m_{31}m_{32} = 0, \tag{7}$$

$$m_{01}m_{03} - m_{11}m_{13} - m_{21}m_{23} - m_{31}m_{33} = 0, \tag{8}$$

$$m_{02}m_{03} - m_{12}m_{13} - m_{22}m_{23} - m_{32}m_{33} = 0. \tag{9}$$

In this work, it will be derived the nine bilinear constraints existing between the 16 elements of the Mueller–Jones matrix by applying the degree of polarization concept directly to a previously determined (theoretical, numerical, or experimental), physically realizable Mueller matrix. In a parallel way to that employed by Hovenier and coauthors [20], the nine bilinear constraints are obtained also by determining the degree of polarization associated with nine outgoing beams of light totally polarized emerging from an optical system.

2. Mathematical Model

The degree of polarization, DoP, associated to a beam of light described by a Stokes vector has been defined by [1]

$$0 \le \mathrm{DoP} = \frac{\sqrt{(s_1)^2 + (s_2)^2 + (s_3)^2}}{s_0} \le 1, \quad S = \begin{pmatrix} s_0 \\ s_1 \\ s_2 \\ s_3 \end{pmatrix}, \tag{10}$$

where s_k, $k = 0, 1, 2, 3$, are the real parameters of the 4×1 column Stokes vector, S, which represents the state of polarization of the light under study. The DoP describes three different possibilities for the measured beam of light: it is totally depolarized if DoP $= 0$; if $0 < \mathrm{DoP} < 1$, light is partially polarized (or partially depolarized, we could say), and it is totally polarized when DoP $= 1$.

The DoP can also be associated with the Mueller matrix, M, of a system by considering the linear response to a beam of light with an incident polarization state, S^i, represented by the well known relation

$$S^o = MS^i \Rightarrow \begin{pmatrix} s_0{}^o \\ s_1{}^o \\ s_2{}^o \\ s_3{}^o \end{pmatrix} = \begin{bmatrix} m_{00} & m_{01} & m_{02} & m_{03} \\ m_{10} & m_{11} & m_{12} & m_{13} \\ m_{20} & m_{21} & m_{22} & m_{23} \\ m_{30} & m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{pmatrix} s_0{}^i \\ s_1{}^i \\ s_2{}^i \\ s_3{}^i \end{pmatrix} = \begin{pmatrix} m_{00}s_0{}^i + m_{01}s_1{}^i + m_{02}s_2{}^i + m_{03}s_3{}^i \\ m_{10}s_0{}^i + m_{11}s_1{}^i + m_{12}s_2{}^i + m_{13}s_3{}^i \\ m_{20}s_0{}^i + m_{21}s_1{}^i + m_{22}s_2{}^i + m_{23}s_3{}^i \\ m_{30}s_0{}^i + m_{31}s_1{}^i + m_{32}s_2{}^i + m_{33}s_3{}^i \end{pmatrix}, \tag{11}$$

where M is a matrix of 4×4 real elements, and S^o is the Stokes vector representing the outgoing beam of light. This means there are required 16 real elements for the complete determination of M. The Stokes elements are related to the orthogonal components, parallel and perpendicular to the plane of incidence, respectively, of the electric field (E_p, E_s), by

$$s_0 = \langle E_p E_p{}^* \rangle + \langle E_s E_s{}^* \rangle, \qquad s_1 = \langle E_p E_p{}^* \rangle - \langle E_s E_s{}^* \rangle,$$

$$s_2 = \langle E_p E_s{}^* \rangle + \langle E_s E_p{}^* \rangle, \qquad s_3 = i(\langle E_p E_s{}^* \rangle - \langle E_s E_p{}^* \rangle), \tag{12}$$

where (*) represents the complex-conjugate operation and the angular brackets indicate a temporal average. From Eq. (12) it can be seen that s_0 represents the total intensity, s_1 represents the magnitude of the p/s component of the light, s_2 represents the tendency to the linear $+45/-45$ $(+/-)$ degree polarization magnitude of light, and s_3 is related to the magnitude of the right-/left-handed (r/l) circular polarization component of light [9].

On the other hand, the linear response of an optical system to incident light can also be interpreted in terms of amplitudes and phases of the electric field. Particularly, if the system does not depolarize, the interaction can be described in terms of the Jones matricial formalism [16]:

$$E^o = JE^i \Rightarrow \begin{pmatrix} E_p{}^o \\ E_s{}^o \end{pmatrix} = \begin{bmatrix} j_{11} & j_{12} \\ j_{21} & j_{22} \end{bmatrix} \begin{pmatrix} E_p{}^i \\ E_s{}^i \end{pmatrix}, \tag{13}$$

where J is called the Jones matrix of the system and the quantities under consideration are complex in general. This means there are implied four amplitudes and four absolute phases for the entire process.

Any system able to be described by a Jones matrix can be described by a Mueller matrix also; however, not any system able to be described by a Mueller matrix can be described by a Jones matrix. Only nondepolarizing systems can be described by both Jones and Mueller matrices, and in this case the Mueller matrix is named the Mueller–Jones matrix. The relation between the Jones matrix and the Mueller matrix is given by [15]

$$M_J = A(J \otimes J^*)A^{-1}; \qquad A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & i & -i & 0 \end{bmatrix};$$

$$A^{-1} = \frac{1}{2}(A^*)^T, \tag{14}$$

where $\otimes$ denotes the Kronecker product and T the transpose operation. Observe that this operation implies the lost of the absolute values for the phases of the Jones matrix and the conversion to relative phases, with the loss of one degree of freedom. This means the Mueller–Jones matrix elements are constituted by products among four amplitudes and three differences of phase obtained from the transformed Jones matrix. In other words, the 16 elements of the Mueller–Jones matrix can be reduced to seven independent parameters only.

The nine bilinear constraints have been obtained also by equating the matrix elements of both sides of the following relationship [15,18,22]:

$$M^T G M = |\det(J)|^2 G = (\det(M))^{1/2} G,$$

where

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

is the Lorentz metric matrix [15,18,22].

3. Mathematical Procedure

In the following, it is shown the nine bilinear constraints between the 16 elements of the Mueller–Jones matrices can be obtained from the degree of polarization definition, DoP, Eq. (10). This procedure is based in the decomposition of a Mueller matrix of an arbitrary optical system. Consider the general Mueller matrix is given by

$$M \equiv M_n + M_d, \tag{15}$$

where

$$M_d = \begin{bmatrix} m_{00} & m_{01} & m_{02} & m_{03} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$M_n = \begin{bmatrix} 0 & 0 & 0 & 0 \\ m_{10} & m_{11} & m_{12} & m_{13} \\ m_{20} & m_{21} & m_{22} & m_{23} \\ m_{30} & m_{31} & m_{32} & m_{33} \end{bmatrix}. \tag{16}$$

The degree of polarization, Eq. (10), and the linear response of a system to the incident light, Eq. (11), can be written as

$$\mathrm{DoP}(M,\ S) = \frac{\left[\sum_{j=1}^{3} (m_{j0}s_0{}^i + m_{j1}s_1{}^i + m_{j2}s_2{}^i + m_{j3}s_3{}^i)^2\right]^{1/2}}{m_{00}s_0{}^i + m_{01}s_1{}^i + m_{02}s_2{}^i + m_{03}s_3{}^i}$$

$$= \frac{\{(M_n S^i)^T (M_n S^i)\}^{1/2}}{\{(M_d S^i)^T (M_d S^i)\}^{1/2}}, \tag{17}$$

$$\Rightarrow \mathrm{DoP}(M,\ S) = \frac{\{(M_n S^i)^T (M_n S^i)\}^{1/2}}{\{(M_d S^i)^T (M_d S^i)\}^{1/2}} \le 1, \tag{18}$$

$$\Rightarrow (S^i)^T [M_d{}^T M_d](S^i) \ge (S^i)^T [M_n{}^T M_n](S^i), \tag{19}$$

where only positive roots have been considered by physical reasons. From the inner $\lfloor M_{d,n}{}^T M_{d,n} \rfloor$ matricial products, Eq. (19), it does follow that

$$m_{00}{}^2 \ge m_{10}{}^2 + m_{20}{}^2 + m_{30}{}^2,$$

$$m_{01}{}^2 \ge m_{11}{}^2 + m_{21}{}^2 + m_{31}{}^2,$$

$$m_{02}{}^2 \geq m_{12}{}^2 + m_{22}{}^2 + m_{32}{}^2,$$

$$m_{03}{}^2 \geq m_{13}{}^2 + m_{23}{}^2 + m_{33}{}^2,$$

$$m_{00}m_{01} \geq m_{10}m_{11} + m_{20}m_{21} + m_{30}m_{31},$$

$$m_{00}m_{02} \geq m_{10}m_{12} + m_{20}m_{22} + m_{30}m_{32},$$

$$m_{00}m_{03} \geq m_{10}m_{13} + m_{20}m_{23} + m_{30}m_{33},$$

$$m_{01}m_{02} \geq m_{11}m_{12} + m_{21}m_{22} + m_{31}m_{32},$$

$$m_{01}m_{03} \geq m_{11}m_{13} + m_{21}m_{23} + m_{31}m_{33},$$

$$m_{02}m_{03} \geq m_{12}m_{13} + m_{22}m_{23} + m_{32}m_{33}. \qquad (20)$$

If the system is considered as a nondepolarizing system, the equality holds in Eq. (20), and the nine bilinear constraints between the 16 elements of the Mueller matrix (a Mueller–Jones matrix indeed) are obtained directly from our procedure. The first term, $m_{00}{}^2 - m_{10}{}^2 - m_{20}{}^2 - m_{30}{}^2 = 0$, must be added to the last three quadratic terms of Eq. (20) in order to reach the nine bilinear constraints, Eqs. (1)–(9).

Note that our procedure is based on the existence of a previously (theoretical, numerical, or experimental) determined and physically realizable Mueller matrix. A question arises: Can we build the nine bilinear constraints by directly measuring the degree of polarization from an outgoing beam of light from an optical system? Let us do the following exercise. By considering a beam of light associated to the six basic polarized Stokes vectors (p, s, $+$, $-$, r, l), incident on a general optical system and by measuring the corresponding degree of polarization outputs, using Eqs. (10) and (11), we obtain the following relations.

The DoP for p-incident polarization is given by

$$(S^o)_p = \begin{bmatrix} m_{00} & m_{01} & m_{02} & m_{03} \\ m_{10} & m_{11} & m_{12} & m_{13} \\ m_{20} & m_{21} & m_{22} & m_{23} \\ m_{30} & m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} m_{00} + m_{01} \\ m_{10} + m_{11} \\ m_{20} + m_{21} \\ m_{30} + m_{31} \end{pmatrix}$$

$$\Rightarrow 0 \leq (\mathrm{DoP})_p = \frac{\sqrt{(m_{10} + m_{11})^2 + (m_{20} + m_{21})^2 + (m_{30} + m_{31})^2}}{m_{00} + m_{01}} \leq 1.$$

From which follows, for p-incident light,

$$(m_{00} + m_{01})^2 \geq (m_{10} + m_{11})^2 + (m_{20} + m_{21})^2 + (m_{30} + m_{31})^2. \qquad (21)$$

Proceeding in a similar way, it can be obtained for the rest of the inputs, the derived DoP relations for s-incident light,

$$(m_{00} - m_{01})^2 \geq (m_{10} - m_{11})^2 + (m_{20} - m_{21})^2 + (m_{30} - m_{31})^2, \qquad (22)$$

for $+45°$-incident light,

$$(m_{00} + m_{02})^2 \geq (m_{10} + m_{12})^2 + (m_{20} + m_{22})^2 + (m_{30} + m_{32})^2, \qquad (23)$$

for $-45°$-incident light,

$$(m_{00} - m_{02})^2 \geq (m_{10} - m_{12})^2 + (m_{20} - m_{22})^2 + (m_{30} - m_{32})^2, \qquad (24)$$

for r-incident light,

$$(m_{00} + m_{03})^2 \geq (m_{10} + m_{13})^2 + (m_{20} + m_{23})^2 + (m_{30} + m_{33})^2, \qquad (25)$$

for l-incident light,

$$(m_{00} - m_{03})^2 \geq (m_{10} - m_{13})^2 + (m_{20} - m_{23})^2 + (m_{30} - m_{33})^2. \qquad (26)$$

By inspection, it can be deduced that Eqs. (21)–(26) contain all the information available from magnitude of light measurements for the six basic polarization states considered here (p, s, $+$, $-$, r, l). By adding and resting by pairs, Eqs. (21) and (22), Eqs. (23) and (24), and Eqs. (25) and (26), respectively, and by considering the equality holds, it can be obtained the following relationships:

$$m_{01}{}^2 - m_{11}{}^2 - m_{21}{}^2 - m_{31}{}^2 + m_{00}{}^2 - m_{10}{}^2 - m_{20}{}^2 - m_{30}{}^2 = 0, \qquad (1)$$

$$m_{02}{}^2 - m_{12}{}^2 - m_{22}{}^2 - m_{32}{}^2 + m_{00}{}^2 - m_{10}{}^2 - m_{20}{}^2 - m_{30}{}^2 = 0, \qquad (2)$$

$$m_{03}{}^2 - m_{13}{}^2 - m_{23}{}^2 - m_{33}{}^2 + m_{00}{}^2 - m_{10}{}^2 - m_{20}{}^2 - m_{30}{}^2 = 0, \qquad (3)$$

$$m_{00}m_{01} - m_{10}m_{11} - m_{20}m_{21} - m_{30}m_{31} = 0, \qquad (4)$$

$$m_{00}m_{02} - m_{10}m_{12} - m_{20}m_{22} - m_{30}m_{32} = 0, \qquad (5)$$

$$m_{00}m_{03} - m_{10}m_{13} - m_{20}m_{23} - m_{30}m_{33} = 0. \qquad (6)$$

Similarly, employing a beam of light with a Stokes vector given by any polarization state of the form $(1 \quad \pm 1/\sqrt{2} \quad \pm 1/\sqrt{2} \quad 0)^T$ and using Eqs. (1) and (2) and Eqs. (4) and (5), Eq. (7) is obtained. Considering an incident beam of light with Stokes vector given by $(1 \quad \pm 1/\sqrt{2} \quad 0 \quad \pm 1/\sqrt{2})^T$ and $(1 \quad 0 \quad \pm 1/\sqrt{2} \quad \pm 1/\sqrt{2})^T$, respectively, and by using the previously determined bilinear constraints, Eqs. (1)–(7), Eqs. (8) and (9) are obtained. The latter three Stokes vectors lie in the planes (s_1, s_2), (s_1, s_3), and (s_2, s_3) including the axis, respectively, of the Poincaré sphere. Observe that if an incident beam with associated Stokes vector of the form $(1 \quad \pm 1/\sqrt{3} \quad \pm 1/\sqrt{3} \quad \pm 1/\sqrt{3})^T$ is considered, the degree of polarization does not provide additional information to Eqs. (1)–(9). In this procedure, it has been employed nine specific polarization states in order to obtain the nine bilinear constraints between the 16 elements of the Mueller–Jones matrix, considering the existence of a physically realizable Mueller matrix. Implicit is the assumption that these input Stokes vectors result in physically realizable output Stokes vectors. For the case of a specific given Mueller matrix, a usual procedure is just to scan for all the possible incident Stokes vectors whose outputs can be associated with physically realizable Stokes vectors (overpolarization condition) [15,22], jointly with a scanning of the gain for all the incident Stokes vectors taken from the Poincaré sphere (overgain condition) [15,22]. These conditions can be plotted in three dimensions as a function of the incident state of polarization parametrized by the angles $0 \leq \psi_i \leq \pi$ (azimuth) and $-\pi/4 \leq \chi_i \leq \pi/4$ (ellipticity) of the polarization ellipse of the wave, respectively [15,22].

Hovenier and coauthors [20] and Hovenier [21], have claimed the set of nine bilinear constrains, Eqs. (1)–(9), is not complete [20,21]. Instead, assuming $j_{22}j_{22}^* \neq 0$, Eq. (13), or equivalently $m_{00} + m_{11} - m_{01} - m_{10} \neq 0$, they have proposed two complete sets each of nine equations [20,21]. In the first work, the complete set has been reported as [20]

$$(m_{22} + m_{33})^2 + (m_{23} - m_{32})^2 = (m_{00} + m_{11})^2 - (m_{01} + m_{10})^2, \tag{27}$$

$$(m_{20} - m_{21})^2 + (m_{30} - m_{31})^2 = (m_{00} - m_{01})^2 - (m_{11} - m_{10})^2, \tag{28}$$

$$(m_{02} - m_{12})^2 + (m_{03} - m_{13})^2 = (m_{00} - m_{10})^2 - (m_{11} - m_{01})^2, \tag{29}$$

$$(m_{02} + m_{12})(m_{00} + m_{11} - m_{01} - m_{10}) = (m_{22} + m_{33}) \times (m_{20} - m_{21}) - (m_{23} - m_{32})(m_{30} - m_{31}), \tag{30}$$

$$(m_{22} - m_{33})(m_{00} + m_{11} - m_{01} - m_{10}) = (m_{02} - m_{12}) \times (m_{20} - m_{21}) + (m_{03} - m_{13})(m_{30} - m_{31}), \tag{31}$$

$$(m_{20} + m_{21})(m_{00} + m_{11} - m_{01} - m_{10}) = (m_{22} + m_{33}) \times (m_{02} - m_{12}) + (m_{23} - m_{32})(m_{03} - m_{13}), \tag{32}$$

$$(m_{23} + m_{32})(m_{00} + m_{11} - m_{01} - m_{10}) = (m_{03} - m_{13}) \times (m_{20} - m_{21}) + (m_{02} - m_{12})(m_{30} - m_{31}), \tag{33}$$

$$(m_{03} + m_{13})(m_{00} + m_{11} - m_{01} - m_{10}) = (m_{22} + m_{33}) \times (m_{30} - m_{31}) + (m_{23} - m_{32})(m_{20} - m_{21}), \tag{34}$$

$$(m_{30} + m_{31})(m_{00} + m_{11} - m_{01} - m_{10}) = (m_{22} + m_{33}) \times (m_{03} - m_{13}) - (m_{23} - m_{32})(m_{02} - m_{12}). \tag{35}$$

These equations were denoted as Eqs. (100)–(102), (106^+), (108^+), (110^+), (112^+), (114^+), and (116^+) in the original paper [20]. The second work reported by Hovenier [21] as the complete set, for the same condition on the Jones matrix, is given by Eqs. (27)–(31) and by the following equations:

$$(m_{22} - m_{33})(m_{00} + m_{11} - m_{01} - m_{10}) = (m_{20} - m_{21}) \times (m_{02} - m_{12}) - (m_{30} - m_{31})(m_{03} - m_{13}), \tag{36}$$

$$(m_{03} + m_{13})(m_{00} + m_{11} - m_{01} - m_{10}) = (m_{20} - m_{21}) \times (m_{23} - m_{32}) + (m_{30} - m_{31})(m_{22} + m_{33}), \tag{37}$$

$$(m_{20} + m_{21})(m_{00} + m_{11} - m_{01} - m_{10}) = (m_{22} + m_{33}) \times (m_{02} - m_{12}) + (m_{23} - m_{32})(m_{03} - m_{13}), \tag{38}$$

$$(m_{30} + m_{31})(m_{00} + m_{11} - m_{01} - m_{10}) = (m_{22} + m_{33}) \times (m_{03} - m_{13}) - (m_{23} - m_{32})(m_{02} - m_{12}). \tag{39}$$

As a matter to show the nine bilinear constraints are not complete, Eqs. (1)–(9), authors have written [20]. "However, a thorough analysis of the three sets of 9 equations published by Abhyankar and Fymat (1969) and by Fry and Kattawar (1981) shows that none of these sets is complete. This may be verified, for example, by observing that the elements of the matrix

$$\begin{bmatrix} 4 & 0 & \sqrt{3} & 1 \\ 0 & 0 & \sqrt{3} & -3 \\ 2 & 0 & -2\sqrt{3} & -2 \\ 0 & 2\sqrt{3} & 0 & 0 \end{bmatrix} \tag{40}$$

obey each one of their three sets of equations but do not satisfy our Eqs. (106^+), (108^-), (110^+), (114^+) and (116^+), nor, e.g., Eq. (152)."

Unfortunately, matrix (40) also does not satisfy Eq. (5) and Eq. (6), which means this is not a good example to verify that the nine bilinear constraints, Eqs. (1)–(9), do not form a complete set. By using the gain as a criterion to verify the physical realizability of a Mueller matrix [15,22] associated with a passive system, it can be easily verified that it is greater than the unity for the matrix considered by Hovenier and coauthors [20]. In other words, the matrix they have considered is not physically realizable. Then, it is not a surprise that the matrix also does not satisfy the set of equations defined by Hovenier and coauthors, Eqs. (27)–(35) [20]. The matrix also does not satisfy Eqs. (36)–(39), as can be easily demonstrated.

If the overpolarization [15,22] and the overgain [15,22] conditions are applied to the matrix considered previously, Figs. 1(a) and 1(b) are obtained for the respective conditions. The maximum value for the degree of polarization is 2.5604 and 5.8013 for the gain, respectively. These criteria confirm the matrix is not physically realizable.

Searching for an optical system able to be described through a Mueller–Jones matrix, in order to test the

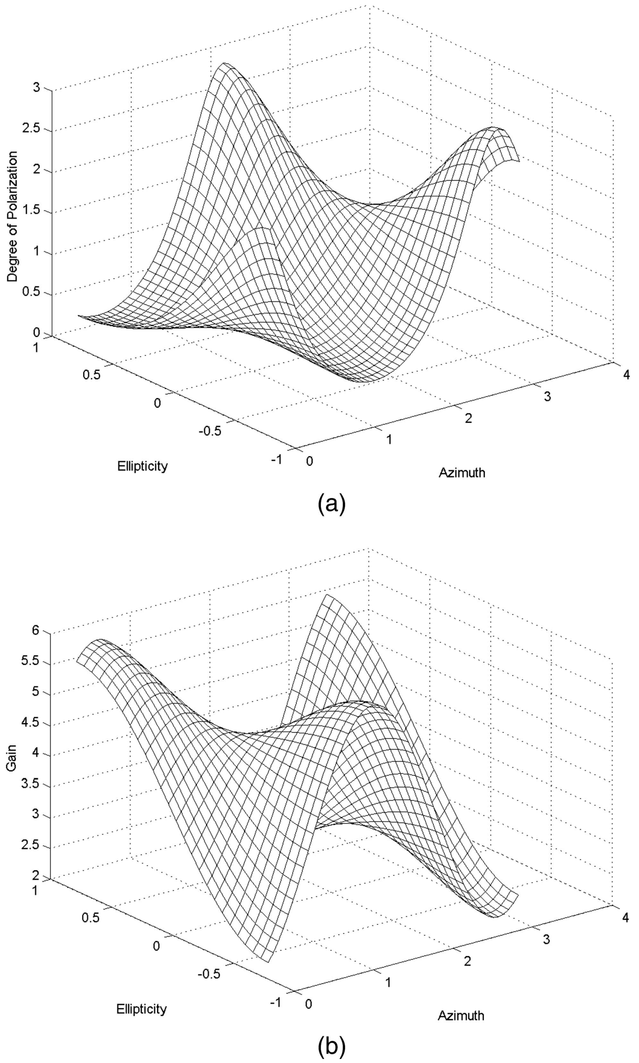


Fig. 1. (a) Output degree of polarization of the optical system described by Eq. (40) as a function of the incident state of polarization parametrized by the ellipsometric angles χ_i and ψ_i. (b) Plot of the gain of the optical system described by Eq. (40) as a function of the incident state of polarization parametrized by the ellipsometric angles χ_i and ψ_i.

validity of the set of Eqs. (1)–(9) and the sets reported by Hovenier and coauthors [20] and by Hovenier [21], respectively, the following situation was found for a retarder (a nondepolarizing system, in principle). Goldstein [3,23] has reported the following Mueller matrices for "the case of a quartz plate that has its optic axis misaligned from the optical axis, inducing a small birefringence." The measured matrix was

$$\begin{bmatrix} 1.000 & 0.019 & 0.021 & -0.130 \\ -0.024 & -0.731 & -0.726 & 0.005 \\ 0.008 & 0.673 & -0.688 & -0.351 \\ -0.009 & 0.259 & -0.247 & 0.965 \end{bmatrix}. \tag{41}$$

After this matrix has been filtered, using an eigenvalue criterium [23], the Mueller matrix reported by Goldstein [23] is given as

$$\begin{bmatrix} 0.737 & -0.005 & 0.006 & -0.067 \\ -0.005 & -0.987 & -0.024 & 0.131 \\ 0.006 & -0.024 & -0.989 & -0.304 \\ -0.067 & 0.131 & -0.304 & 0.674 \end{bmatrix}. \tag{42}$$

Applying the overpolarization and the overgain conditions to both Eqs. (41) and (42) results in Figs. 2(a) and 2(b) and 3(a) and 3(b), respectively. The maximum degree of polarization (gain) is 1.168 (1.133) for Eq. (41) and 1.440 (0.804) for Eq. (42), respectively. Based on this criterium, both matrices are not physically realizable also.

There exist other criteria for the physical realizability of the Mueller–Jones matrices [2,3,15,24–27], and the interested reader can consult the appropriate

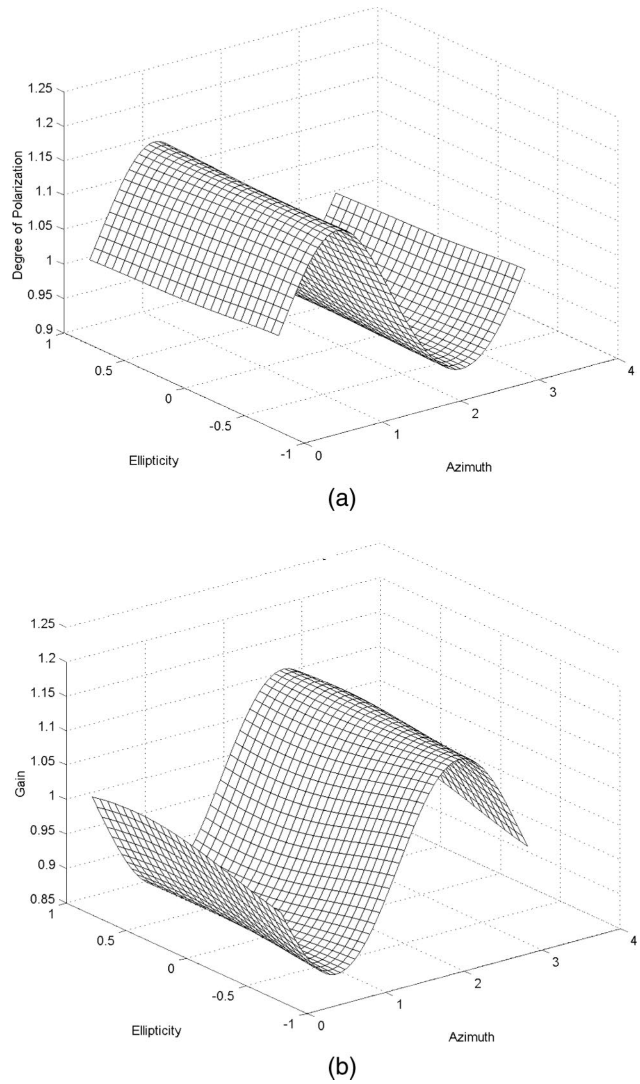
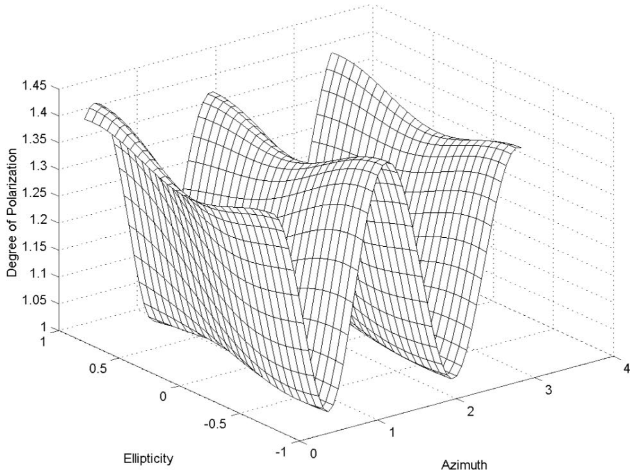
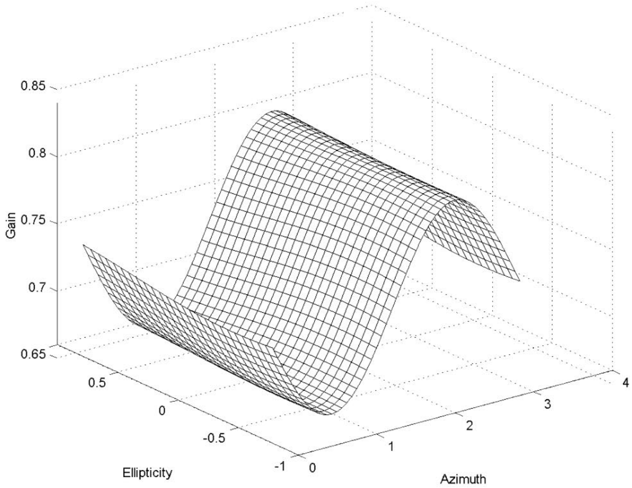


Fig. 2. (a) Output degree of polarization of the optical system described by Eq. (41) as a function of the incident state of polarization parametrized by the ellipsometric angles χ_i and ψ_i. (b) Plot of the gain of the optical system described by Eq. (41) as a function of the incident state of polarization parametrized by the ellipsometric angles χ_i and ψ_i.



(a)



(b)

Fig. 3. (a) Output degree of polarization of the optical system described by Eq. (42) as a function of the incident state of polarization parametrized by the ellipsometric angles χ_i and ψ_i. (b) Plot of the gain of the optical system described by Eq. (42) as a function of the incident state of polarization parametrized by the ellipsometric angles χ_i and ψ_i.

references by using the database of the most important journals concerning this topic, like the references provided in this work among others, or by searching through the Web. To my modest knowledge, it seems all the existing criteria are, in practice, only necessary conditions for the physical realizability of Mueller matrices, and a single necessary and sufficient condition is missing from the state of the art.

4. Conclusions

The degree of polarization operative definition has been employed as a criterion to find the nine independent relations between the 16 elements of the Mueller–Jones matrix. This procedure has been applied by considering a previously determined, physically realizable Mueller matrix. Finally, the nine bilinear constrains have been obtained by directly measuring the degree of polarization from an outgoing beam of light from an optical system by considering the nine totally polarized incident states of light. For practical purposes, all the incident polarization states must be scanned from the Poincaré sphere, in order to satisfy the overpolarization and the overgain conditions, respectively, for the physical realizability of the Mueller matrix.

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